An Experimental Study

TO DETERMINE THE RELATIVE DIFFICULTY OF THE ELEMENTARY NUMBER COMBINATIONS IN ADDITION AND MULTIPLICATION.

"A thesis presented to the Faculty of the Graduate School of the University of Pennsylvania in partial fulfilment of the requirements for the degree of Doctor of Philosophy."

BY

HARRY VANCE HOLLOWAY.

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Contents.

I. THE NEED OF SUCH AN INVESTIGATION.

Hundreds of thousands of children throughout the civilized world are each year confronted with the task of mastering the elementary facts of number. These number facts or processes are the result of centuries of development. They are now an indispensable part of human knowledge. They, therefore, constitute an absolutely unquestioned part of every system of elementary education. Any additional knowledge that may tend to simplify or to facilitate the mastery of these facts must be of value to the great army of teachers upon whom devolves the task of teaching them, and of even more value to the greater army of children whose task it is to learn them. It is helpful for both teacher and pupil to know where the points of difficulty lie, because it stimulates both to more intense activity when these points are reached. This activity finds expression on the part of the teacher in the invention of devices for teaching, in the development of methods of instruction. With the pupil this activity manifests itself under proper guidance in greater attention, in greater effort to get and hold the fact to be acquired or the problem to be mastered. Like one about to perform a feat of strength, he rouses the energy which he thinks may be sufficient to meet the demands of the occasion. We have, therefore, sought by this investigation to determine the relative difficulty of the elementary combinations, recognized generally as such in the United States, of the two most fundamental processes in arithmetic, namely, addition and multiplication. We do not presume that this study is in any way commensurate with the far-reaching application of the problem involved. We can only claim that what follows describes the most comprehensive investigation of this particular field of inquiry that has been made up to the present time.

It is not an uncommon thing in the school room and out, when the combinations are learned by "tables," to find little or no distinction in the emphasis placed upon the various facts of a given table. And the same is sometimes true even among the various tables themselves. Two plus three receives just about as much attention as does $2 + 7$; and 2×1 receives about the same emphasis as does 2×9, though in both cases the difference in difficulty is at once apparent. The most serious objection to this method of procedure is the woeful waste of time involved in repeating again and again facts that need but few repetitions to make their recall automatic. Nor can it be said that the writers of text-books of primary arithmetic have kept the matter of relative difficulty sufficiently in mind. Their chief aim seems to have been a logical development, perhaps a psychological development, of all the elementary number facts in their regular order, and the giving of comprehensive miscellaneous review drills from time to time as the progress of the work would seem to demand. To the credit of a few be it said that they have designated, somewhat indefinitely, to be sure, a few of what might be deemed difficult combinations, but they have, for the most part, laid the burden of determining the emphasis necessary to their mastery upon the teacher. This is doubtless only partly as it should be. The writer of such a text-book should know rather definitely beforehand just where inherent difficulty lies, so that he may present his material accordingly, and the teacher should have some more specific guide than is furnished in the ordinary text as to the place and amount of emphasis necessary for best results. She should know about what results she may reasonably expect in teaching a given group of number facts with a given amount of drill. It is the object of the present investigation to furnish, as far as may be, just this information.

In order to determine just what emphasis was given the various combinations in addition, as shown by frequency of recurrence, the writer made a detailed study of three of the best number primers at present on the market. (See Table I.) These books were written with the idea of being put into the hands of the child at the end of the first two or three weeks of his school experience, and suggesting sufficient material to occupy his attention until about the end of the first half of the second year.

TABLE I.

SHOWING THE NUMBER OF PRESENTATIONS GIVEN TO EACH OF THE ADDITION COMBINATIONS IN THREE OF THE BEST NUMBER PRIMERS.

Combination.	*No. 1.*	*No. 2.*	*No. 3.*
0 + 0	1	76	5
1 + 0	12	24	49
2 + 0	27	47	47
3 + 0	17	33	50
4 + 0	13	29	55
5 + 0	14	50	38
6 + 0	5	31	47
7 + 0	5	30	34
8 + 0	3	38	41
9 + 0	5	26	27
1 + 1	44	88	42
2 + 1	85	140	114
3 + 1	80	72	104
4 + 1	63	60	95
5 + 1	44	58	92
6 + 1	38	41	93
7 + 1	30	55	79
8 + 1	36	62	78
9 + 1	20	24	61
2 + 2	71	47	65
3 + 2	130	114	134
4 + 2	105	71	138
5 + 2	89	75	109
6 + 2	61	75	128
7 + 2	59	53	103
8 + 2	48	41	92
9 + 2	40	26	94
3 + 3	58	46	45
4 + 3	109	72	119
5 + 3	98	75	103
6 + 3	66	60	93
7 + 3	55	38	88
8 + 3	42	21	76
9 + 3	43	30	78
4 + 4	42	33	35
5 + 4	72	45	98
6 + 4	34	36	99
7 + 4	38	20	80
8 + 4	39	23	90
9 + 4	33	18	77
5 + 5	36	47	49
6 + 5	33	29	83
7 + 5	30	24	72
8 + 5	25	27	68
9 + 5	18	20	76
6 + 6	11	11	39
7 + 6	15	27	98
8 + 6	10	17	85
9 + 6	11	15	71
7 + 7	3	8	36
8 + 7	9	19	17
9 + 7	9	19	66
8 + 8	5	7	38
9 + 8	11	16	69
9 + 9	5	9	20
Total Presentations.	2105	2298	3982

In determining the number of presentations of the various combinations, continuous counting by 2's, 3's, 4's and 5's to a certain sum was omitted. Primer No. 1 thus gave 2,105 presentations; No. 2 gave 2,298 presentations, and No. 3 gave 3,982 presentations. A comparison of the corresponding columns of Table I shows an interesting lack of uniformity in the relative number of presentations of the facts involved. If the number of presentations given by the three authors to the various combinations is to be regarded as a measure of the relative difficulty of these combinations, then No. 1 would have us believe that $3 + 2$ is the most difficult fact to fix in the pupil's mind, and that $0 + 0$ is the least difficult. No. 2 would have us believe that $2 + 1$ is the most difficult and needs 140 repetitions during the time given to fix it in the child's mind, while $8 + 8$ is the least difficult and needs but 7 repetitions to accomplish the same result. No. 3 says that $4 + 2$ is the most difficult and will require 138 repetitions, while the child can learn $0 + 0$ in 5 repetitions, and $9 + 9$ in 20 repetitions. Considering the total number of presentations given, we should expect No. 3 to give approximately twice as many repetitions of each combination as No. 1, yet they give $1 + 1$ practically the same number of times, while No. 2 gives twice as many repetitions of this fact as either of the others. No. 2 would indicate that $0 + 0$, $3 + 1$, $4 + 2$, $5 + 2$, $6 + 2$, $4 + 3$, and $5 + 3$ are of practically the same difficulty, while the difference in difficulty in these combinations as shown by No. 1 and No. 3, varies considerably. The fewer repetitions of the double numbers; as, $7 + 7$, with No. 3 and No. 2 may be attributed not to a belief that they may be easier than others, but to the fact that there are no reverse forms, as with $7 + 5$, which also is made to occur as $5 + 7$. There is a double significance in these comparisons: (1) there is no agreement among authors as to the proper number of presentations to be given to the various facts; and (2) the number of repetitions of a given combination by the same author is in no sense a guide to the teacher in determining the relative difficulty of that combination.

II. RECOGNITION IN CURRENT PEDAGOGICAL LITERATURE AND SYLLABI OF THE NEED OF SUCH AN INVESTIGATION.

That there is in this matter of relative difficulty of number combinations in both addition and multiplication a problem worthy of investigation has been pointed out by such specialists in education as Prof. Henry Suzzalo (31), Dr. A. Duncan Yocum (44), and Dr. C. W. Stone (36); and indeed the best teachers of primary number work have for a great many years recognized its existence, and have acted upon their more or less satisfactory opinions and observations in their teaching practice. Certain investigations along this line have even been made, in this country by Phillips (23), Griggs (5) and Phelps (22), and in Germany by Döring (2), the methods and results of which we shall take the liberty of giving and discussing in subsequent pages.[1] Such writers on special method as McMurry (17), Yocum (46), Quigley (26), Rigler (28), and Miss Gildemeister (3), have each made important suggestions arising out of their recognition of a difference in difficulty of either addition or multiplication combinations, or both. So also has one of the most recent state monographs (21) on the teaching of arithmetic this to say: "Much waste of effort will be avoided if these facts (of addition) are separated into groups according to their difficulty," . . . and again, "the difficulties in addition are largely because a few simple combinations are troublesome such as $9 + 7$, $8 + 5$, $8 + 7$, $7 + 4$, $5 + 2$, $5 + 3$, $8 + 5$. The stress of drill should be placed upon these and other points of difficulty." Doubtless much other evidence might be found to show a still wider recognition of the existence of an important problem here, but that which is cited above seems quite sufficient for our purpose.

[1] P. Ranschburg (27), of Budapest, in his articles zur physiologischen und pathologischen Psychologie der elementaren Rechenarten, referred to by Ernest C. McDougle (15) in his excellent summary of studies made in arithmetic as treating the problem of "relative difficulty," deals with the relative difficulty of the four fundamental processes, comparing the results obtained from normal and backward children.

III. WHAT HAS BEEN DONE TOWARD THE SOLUTION OF THE PROBLEM, AND THE METHODS OF ATTACK.

1. THE WORK AND RESULTS OF PHILLIPS.

The earliest attempt reported in this country to gather any systematic information along the line of our inquiry was made in 1897 by Mr. D. E. Phillips (23), then of Clark University. Mr. Phillips sent out a syllabus to 616 persons, 72% of whom were teachers, and the other 28% men and women whose occupations are not designated. Among a number of other questions this was asked: "What numbers give most trouble in adding or multiplying?" There were 440 returns received, 157 of which gave 7 and 9; 88, 7 and 8; 34, 6 and 7; 42, 7 only; 26, 3, 6 and 8. The others were miscellaneous. Seven is found in 327 cases, 9 in 204. Five say that 9 is easy, always one less than 10. In addition to this, the errors in multiplication found in the papers of 283 ninth grade pupils involving 1,095 problems or multiplications, the majority of which contained three figures in the multiplier and four in the multiplicand, were tabulated and are worthy of note. There were in all 691 mistakes in multiplication. Of these 186 were made in multiplying by 9; 195 by 8; 199 by 7; 57 by 6; 9 by 5; 15 by 4; 3 by 2. (24).

2. THE WORK AND RESULTS OF GRIGGS.

Mr. A. O. Griggs (5), also of Clark University, in an article published in the Pedagogical Seminary, September, 1912, entitled "The Pedagogy of Mathematics," has done us the service of bringing together the work of Mr. Phillips and that of Professor Döring. Where there is a difference of results, he allows those of Professor Döring the preference for reasons that will appear in the following summary of the latter's experiments.

3. THE WORK AND RESULTS OF DÖRING.

Prof. Max Döring's investigation (2) is one of two very satisfactory pieces of work that have been done up to the present time to throw more light on this particular problem.[1] His study involved only the multiplication combinations up to 10×10. He started out by asking a number of adults (mostly teachers, both men and women) to name the three elementary multiplication combinations which in the opinion of each, based upon his own private and his pedagogical experience, were the most difficult. The results were so surprisingly uniform that he extended the inquiry to 323 boys in age from nine to fourteen years. To each of these the same question was put in writing, and after some deliberation each wrote down his opinion. To avoid the possibility of their hesitating to put down their real difficulties, they were not required to write the answers to the combinations which they might give.

In compiling the results, the numbers with their reverses, as 7×8 and 8×7, were tabulated under the same heading of 8×7 as one fact. Sixteen combinations appear in 880 out of the 1,053 opinions given. Sixteen and four-tenths per cent. of the opinions were eliminated as being unworthy of statistical expression. Forty-nine per cent. of the opinions indicated 8×7 to be the most troublesome combination, 34% stood for 9×7, 30% for 9×8, 30% for 7×6, 21% for 9×6, &c. That the selection of the combinations indicated was not a mere chance selection is demonstrated by statistical formulæ for determining the mathematical probability and the possible variation therefrom.

From the sixteen combinations, Professor Döring next proceeded to arrange the various tables in the order of difficulty suggested thereby. Noting that the factor 7 appears most fre-

[1] C. L. Phelps of Leland Stanford University (22) in 1913 worked over from a different point of view material collected by Otis and Davidson ["The Reliability of Standard Scores in Adding Ability," El. School Teacher, Oct., 1912] from 238 eighth grade pupils to determine the frequency of error in addition combinations. For his results partially stated, see Table VIII.

quently in the opinions tabulated, he concluded, as Dr. McMurry also states (18), that the seventh row is the most difficult. For the sake of comparison we give below the order of difficulty thus obtained side by side with that suggested by Dr. McMurry, Miss Gildemeister (4), and Professors Rocke, Roegers and Wolf (13). The most difficult tables are placed on the left, the least difficult on the right of the table.

TABLE II. (a).

Döring	7	8	9	6	4	5	3	10	2	1
McMurry	7	9	6	3	8	4	5	2	10	..
Gildemeister (omitting 0, 11 and 12)	8	7	6	4	3	5	9	10	1	2
Rocke, Roeger & Wolf	7	9	6	3	8	4	5	10	2	1

This is indeed an interesting comparison of opinion and experience with the simplest possible method of statistical investigation. Professor Döring recognizes three general orders of difficulty: the 1st, 2d and 10th tables he regards as "leicht"; the 3d, 5th and 4th, as "mittel schwer"; and the 6th, 9th, 8th and 7th, "schwer."

In order to check up his first results this investigator selected the ten most difficult combinations as determined by his first experiment and gave them as a test to 100 boys of the third school year. Among the 9,600 answers appeared 175 errors, or 1.8% distributed as follows:

TABLE II. (b).

New Place	1	2	3	4	5	6	7	8	9	10
Combination,	9×7	9×6	8×8	8×7	8×6	7×6	9×8	7×7	8×4	9×9
No. Errors	25	25	23	21	21	17	14	13	12	4
Old Place	2	5	7	1	6	4	3	8	10	9

The number of errors is, of course, so small as to deprive this part of the experiment of any conclusive results, yet it is interesting here to note the comparatively small divergence from the first order given, and this is all the more interesting when it is noted that special search is made in the third year of the German schools for the difficult combinations, and special drill is given upon them.

IV. AN ESTIMATE OF THESE INVESTIGATIONS—THEIR LIMITATIONS IN SCOPE AND METHOD.

If nothing else were accomplished by the contributions described above than to call the attention of the teaching profession to the existence here of an important factor in the acquisition of number facts, they would even then be eminently worth while. But they have done more. In the first place they have suggested methods of attacking other problems of no less importance in school-room practice. In the second place they have shown that, while empiricism may be a reliable guide to such practice, it should be verified by careful experiment and receive the sanction of scientific investigation. Only thus can certainty and dignity be given to method. And, lastly, they have contributed information of real value in acquiring mastery over a very essential part of the school curriculum.

If these things are true, how shall we justify further investigation along the lines indicated? It will, in the first place, be noted that the relative difficulty of the addition combinations is scarcely touched upon, except for eighth grade pupils (see footnote, page 13), while practically only sixteen of the multiplication combinations are statistically considered. The limitations first noted are justified in the light of the larger subject of "Number and its Application," discussed by Mr. Phillips (23), so likewise with that of "The Pedagogy of Mathematics," discussed by Mr. Griggs(5). But the limitations of Professor Döring's study were set by himself rather than by his subject to the "difficult cases," though his subject, "Zur Psychologie des Kleinen Einmaleins," would suggest a somewhat more comprehensive view of the psychological side of the problem than he gives.

Aside from the limited scope of the studies made, it is doubtful if the questionnaire method with adults or the "method of errors" in problems involving many figures in the multiplicand and multiplier, where some of the errors may be errors in "carrying," are greatly to be relied upon. So likewise with a method which depends largely upon the "opinions" of those tested. The data obtained by the use of such methods should at least be checked up by results obtained by methods not open to the objections that may be offered to any of the above mentioned modes of procedure.

V. THE PURPOSE OF THIS STUDY.

It is the purpose of this study to determine, as nearly as may be, the relative difficulty of certain number facts, and to express the data obtained in such a way that primary teachers may be furnished a more definite guide than now exists as to the emphasis that should be placed upon these facts in their attempt to impart them to the child. It will also be a part of our task to suggest some of the elements of method which may be profitably used to reach the ends of instruction in this field in a more expeditious way than has formerly been done.

VI. THE PLAN OF THE INVESTIGATION.

The teaching of both the addition and the multiplication combinations requires, in the nature of the case, the breaking up of the work into certain groups, which must consist of fewer facts than are given in a single "table." It is, therefore, important that the teacher know the points of difficulty in each of the several groups as they are considered. In other words, it is important that a survey of the whole field of elementary number facts be made rather than only the mountainous districts. Small climbs to the untrained beginner may tax his energies quite as much as the more inaccessible peaks do to him who by training and experience has learned the art of the mountain climber, and has developed in the process the physical powers necessary to the bigger task.

1. DIFFICULTY AS SHOWN IN LEARNING THE COMBINATIONS.

Since children have to master the facts as presented, it has been our plan to study the difficulties which the children themselves have experienced in the process. This, though probably the most difficult plan of attack—surely it has been the most tedious—has seemed to us to be the most logical. It is the child's point of view that must in the end be most helpful for the teacher to get.

2. DIFFICULTY AS SHOWN BY ERRORS MADE IN THE COMBINATIONS.

We have also attacked the problem from another side. We have determined the relative number of errors made by large numbers of children in writing the combinations after these children have been over the ground and were presumably familiar with them. This plan differs from Professor Döring's second method of determining his results only in the fact that we have in our experiments provided equal opportunity to make errors in all of the combinations, while he gave tests on only the ten combinations which he found most difficult in his first experiment.

3. DIFFICULTY AS SHOWN BY THE COMBINATIONS FORGOTTEN.

It also seemed wise to consider the relation between the various number facts from the standpoint of their permanence when once mastered. It is not infrequently the case that things easily learned are quite as easily forgotten. With this point in mind we have given tests in all the number facts to over 500 third grade children during the last week of their school work in June, and have given the same tests to the same children in the first week of their school work in September following. We have thus determined the facts which are most likely to drop out of memory during the long summer vacation. This should be of interest to a fourth grade teacher just taking a class promoted from the third grade.

4. DIFFICULTY AS SHOWN BY THE TIME REQUIRED TO WRITE THE COMBINATIONS.

And finally, since the more difficult a mental process is, the more thought it requires, the less skilful the individual is in its use, the greater is the time required in its accomplishment. It has, therefore, been also our plan to determine the relative difficulty of the various groups of facts as wholes by comparing the length of time required to write correct results to the various combinations of which these groups were constituted. These tests, like that described immediately above, were given only to pupils who had been taught all the facts involved.

VII. THE LIMITS OF THE FIELD OF INVESTIGATION.

1. THE TWO FUNDAMENTAL PROCESSES.

Of the "four fundamental processes" in arithmetic, two may be regarded, in the light of modern methods, as fundamental, and the other two derivative. Subtraction may be regarded as addition from another point of view, as far as the process goes, if the Austrian or making change method is used, and division is likewise multiplication from another point of view. In the final analysis there is really only one fundamental process, viz., addition, all the others being derived from this. While multiplication may be regarded as "shorthand" addition, and so taught, yet it is not the addition of a row containing seven 4's that a child should see when he says seven 4's are 28. The latter is purely an abstract process, entirely different psychologically from adding 4 and 4 and 4, &c., seven times. It is quite in accord with the best established pedagogical principles that the known process of addition be carried over and made the basis of its modification in multiplication, but its *raison d'être* having been once pointed out the connection should cease as far as the process goes. During the process of multiplication introspection shows no conscious association with the process of addition. This is not true in the case of subtraction, except when it has been taught as a separate process, nor is it the case with division, there being in the first case a conscious background of addition, and in the second a conscious background of multiplication. Four from twelve leaves eight, because eight and four are twelve; four into twelve goes three times, because three times four are twelve. For these reasons we have selected what we regard as the two most fundamental processes for the field of our investigation, and we believe that what may be found true in this field may be regarded in a very large measure as true in the related fields of subtraction and division within the same range.

2. THE FORTY-FIVE ADDITION COMBINATIONS.

If all the possible combinations in addition from 0 to 9 are counted, it will be found that the total is 100. Of these, 19 are combinations involving 0. There are then 81 combinations of the 9 significant numerals. Of these, 9 admit of no reverse forms; as, $1 + 1$, $2 + 2$, $3 + 3$, &c. Excluding these 9 there are 72 combinations, counting both direct and reverse forms; as, $4 + 3 = 7$, and $3 + 4 = 7$. If we regard both of these forms as one number fact, there are in these 72 combinations 36 number facts, which, added to the 9 admitting of no reverse form, gives us 45 fundamental addition combinations. In this investigation we have dealt with the 81 combinations in the teaching process for reasons which we shall presently explain, and we have done the same in the "time tests" (See VI 4), but in the final test to determine the number of errors made after the teaching process was finished only the 45 fundamental facts were given. Since the combination of a single number with 0 never occurs in practice—we are never required to add 0 objects to 4 objects or 4 objects to 0 objects—the necessity appearing only in the handling of numbers above 9, as when we say 8 and 10, or 30 and 15, it was though unnecessary to include the 0's in the experiments. All the results in addition are, therefore, subsumed under 45 fundamental heads.

3. THE SEVENTY-EIGHT MULTIPLICATION COMBINATIONS.

In this country it is customary to teach the multiplication tables up to 12×12. It is also generally the custom not to teach the combinations with 0 as a part of the tables, they being left till they are required in column multiplication. The combinations with the ones, however, are taught, at least in the form in which the *one* appears in the second factor, or is used as the multiplicand. We hear, for instance, two 1's, three 1's, four 1's, &c., at the beginning of each table. Not counting the combinations using 0 as a factor, there are 144 combinations from 1×1 to 12×12, inclusive. Of these 12 have no reverse form;

as, 1×1, 2×2, 3×3, &c. Excluding these 12 there are 132 combinations counting both the direct and the reverse forms; as, $7 \times 4 = 28$ and $4 \times 7 = 28$. If we regard both of these as one number fact, there are in these 132 combinations 66 fundamental facts, which, added to the 12 admitting of no reversions, gives us in all 78 fundamental facts in all the multiplication tables. While the multiplication combinations have been taught in both direct and reverse forms for reasons which we shall subsequently adduce, and the "time tests" were made to include both of these forms, as in the case of addition, only 78 facts were required in the final tests, and the results of all were subsumed under the 78 fundamental heads.

4. THE NECESSITY AND ADVANTAGES OF TEACHING BOTH DIRECT AND REVERSE FORMS.

One of the earliest problems which confronted us in conducting this investigation was whether we should take up the "tables" in addition and multiplication in the time honored way, or whether we should reduce the number of combinations to the forty-five fundamental addition facts and the seventy-eight fundamental multiplication facts. Upon the decision on this point depended a most important factor in the method of presenting the work. If it could be shown that a pupil who knew perfectly $4 + 7 = 11$, would know equally well $7 + 4 = 11$, or if it could be shown that a pupil who knew surely $4 \times 9 = 36$, would know with equal certainty $9 \times 4 = 36$, this would entirely eliminate the teaching of all reverse forms.

a. Tests in Addition.—In order to determine what was possibly true in regard to this matter in addition, a test was made with 29 children, 14 boys and 15 girls, in the lower half of the first grade. The experiment began on October 28th and was concluded November 1st, 1912. The children had been in school since the middle of September, and had already learned a few of the simpler addition facts. The following facts in the form indicated were selected for the drill: $6 + 8$, $7 + 6$, $9 + 8$, $8 + 7$, $9 + 6$. These facts were drilled upon with equal oral

and written emphasis from Monday till Thursday, inclusive. On Friday morning at ten o'clock, after sixty minutes in school, a written test was given on the forms as drilled. At eleven-twenty o'clock of the same morning, forty-five minutes after a twenty-minute recess and singing period, the same test was given in the reverse form. The results are given below:

	No. Pupils.	6+8	7+6	9+8	8+7	9+6	Total Correct.	% Correct.
Direct Form......	29	29	29	27	29	27	141	97.2
Reverse Form.....	29	21	25	19	20	20	105	72.3

Out of a possible 145 correct results it will be noted that ten times as many errors were made in the reverse form as in the direct form test.

b. Tests in Multiplication.—At the same time that the experiment in addition was being conducted a similar experiment in multiplication was being carried on with 40 pupils, 20 boys and 20 girls, in the lower half of the second grade. In this experiment there was only one-fourth as much written drill as oral drill. The final test in the direct form was given at ten A. M. on November 1st, and that in the reverse form at eleven o'clock on the same morning, fifteen minutes after a ten-minute recess. The results are given in the following table:

	No. Pupils.	3×4	7×3	6×4	5×3	4×5	Total Correct.	% Correct.
Direct Form......	40	40	39	38	38	38	193	96.5
Reverse Form.....	40	40	33	35	37	35	180	90

The selection of 3×4 as one of the combinations was doubtless not well made on account of its manifest ease, yet even in this experiment nearly three times as many errors were made in the reverse form test as in the direct form test.

c. Conclusion.—From these experiments we concluded that, while most of the direct form teaching could doubtless be carried over and applied to the reverse form, not a sufficient amount was transferred to justify the teaching of one form and omitting to teach its reverse, and since the time of Grube (1842), it may be said that the teaching of both forms has been in accord with the best pedagogical practice. We, therefore, have double justification for the plan pursued in this investigation.

VIII. GROUPING.

1. THE ADDITION COMBINATIONS.

The necessity of teaching both the direct and reverse forms naturally brought up the question as to the order in which they should be taught. This in turn involved still another problem, that of grouping the combinations to be taught. Should we take one table at a time in what might be regarded as the logical order, or should we take a psychological order, such as is pointed out by Dr. Yocum (46) for the addition facts, or one such as is indicated by Superintendent Rigler (28) in the Portland, Oregon, Course of Study, or what may be regarded as the promiscuous order suggested by the Los Angeles Course of Study (14). As none of these is even approximately the same, it was concluded to accept the psychological order suggested by Dr. Yocum's experiment and break his groups into smaller ones or combine them into larger ones containing each five facts, the idea being that one new fact each school day for nine weeks might cover the ground in addition. As a result the following groups with their reverse forms were selected:

1.	1 + 1,	2 + 1,	3 + 1,	4 + 1,	5 + 1
2.	6 + 1,	7 + 1,	8 + 1,	9 + 1,	2 + 2
3.	3 + 2,	4 + 2,	5 + 2,	6 + 2,	7 + 2
4.	8 + 2,	9 + 2,	3 + 3,	4 + 3,	5 + 3
5.	6 + 3,	7 + 3,	8 + 3,	9 + 3,	4 + 4
6.	5 + 4,	6 + 4,	7 + 4,	8 + 4,	9 + 4
7.	5 + 5,	6 + 5,	7 + 5,	8 + 5,	9 + 5
8.	6 + 6,	7 + 6,	8 + 6,	9 + 6,	7 + 7
9.	8 + 7,	9 + 7.	8 + 8,	9 + 8,	9 + 9

2. THE MULTIPLICATION COMBINATIONS.

In multiplication the first nine facts were made to constitute the first group for those who had already had some work in the process, *i. e.*, from 1×1 to 9×1. For those who were beginning the work for the first time these nine facts were divided into two groups, the first of five combinations, the second of four. The remaining combinations were divided into groups of five facts each, in the same order as shown above for the addition

facts, up to the last group, which consisted of only four facts, 12×10, 11×11, 12×11, 12×12. There are thus sixteen groups of multiplication facts, it being the idea that this ground might be covered in sixteen weeks of five days each, thus giving an average of one new fact each day. Experience with this manner of grouping, and the definiteness of the work required made it possible to accomplish the tasks in the time allotted except with the class entering school in the fall term. The corresponding class in the spring term, however, had no difficulty in keeping up with the schedule.

3. ADVANTAGES OF THE GROUPING SELECTED.

The advantages of this manner of grouping are two in number: (1) the general idea of the sequence has already been shown to be psychological (46), at least as far as the addition facts are concerned; and (2) this order following the order of the numerical scale does not lay the investigator open to the charge of a predetermined attitude of mind toward the problem of relative difficulty by a special grouping devised to demonstrate some possible theory of his own.

IX. METHOD OF PROCEDURE IN THE FIRST PART OF THE INVESTIGATION.

1. PRELIMINARY TESTS.

The first direct step in the investigation was made at the beginning of the second term, February 3d, 1913. It consisted in giving preliminary tests in addition to the first grade and the first half of the second, and in multiplication to the second half of the second grade and the third grade. The forty-five addition combinations in promiscuous order were arranged in columns on a sheet of paper in the following form: $2 + 5 =$, $3 + 4 =$, &c., with the answers omitted as indicated. Similarly, the seventy-eight multiplication combinations were arranged in promiscuous order in three general divisions, representing about the first, second and third third's, respectively, of the whole

number of facts, the object being to arrange the test to suit in some degree the various stages of advancement of the pupils tested. The sheets on which the combinations had been mimeographed were distributed face down, one on the desk of each pupil in the various rooms. The teachers then explained that there were a number of little examples in addition or multiplication, as the case might be, on the sheet, and they wished to see how quickly each pupil could write the answers after the equal sign, illustrating on the blackboard just what was expected. The pupils were then told to turn their papers and write. They were permitted to write as many answers as they could. This test furnished the preliminary test record for each pupil for each combination studied until the completion of the drill. The same method was pursued at the beginning of the experiment in September, 1913, except in the first grade, where the pupils could write only with great difficulty, if at all. The results here were obtained by oral examination by the teacher.

This done, each teacher was provided with a list of the groups to be taught as indicated in Section VIII, and a copy of directions as follows:

2. DIRECTIONS TO TEACHERS FOR CONDUCTING DRILLS AND TESTS.

The same amount of time and the same emphasis should be used each day in presenting the groups of combinations to be taught. The objective demonstration should be used only *twice*, and on successive days. The presentation should consist of *ten* (10) oral repetitions, *three* (3) of which should be in concert with the whole body of pupils studying the group, and the other *seven* (7) by sections and individuals. There should be *five* (5) written repetitions of each number fact each day. In all of this work of repetition only the *correct* form is to be allowed.

Subtraction should be taught as the reverse process of addition, and division as the reverse process of multiplication. The direct and the reverse processes should go hand in hand.

The multiplication facts may be taught in the form of tables, but they should appear in both the forms as shown below:

$2 \times 1 =$		$1 \times 2 =$
$2 \times 2 =$		$2 \times 2 =$
$2 \times 3 =$	and	$3 \times 2 =$
$2 \times 4 =$		$4 \times 2 =$
$2 \times 5 =$		$5 \times 2 =$

but all tests should be made *promiscuously*, both when oral and when written.

Teach addition combinations whose sums do not exceed *ten* first by the use of objects, but only *twice* thus.

Teach multiplication as short-hand addition, illustrating by the repetition of the same number in column arrangement to be added.

Drill after each written test until at least 97% of class efficiency is obtained, then proceed to the next group.

Review all combinations finished twice each day, once oral and once written, preferably in connection with the problem work.

Two weeks after giving the final test on each group a review test on that group is to be given along with the daily test on the same sheet of paper, but on the reverse side, such side to be marked "Review."

Read over these instructions carefully and follow them implicitly, as the results will be vitiated by any lack of uniformity in the presentation of the material.

3. PERCEPTION CARDS.

In order to facilitate the drills and tests and at the same time provide an easy means of visual instruction, special perception cards for both addition and multiplication were prepared by the writer in the form indicated in the following figures:

Fig. 1.

$$4 + 7 = 11$$

$$7 + 4 = 11$$

Obverse.

Fig. 2.

$$\begin{array}{r} 4 \\ +\ 7 \\ \hline \end{array}$$

$$\begin{array}{r} 7 \\ +\ 4 \\ \hline \end{array}$$

Reverse.

Fig. 3.

$$4 \times 4 = 16$$

Obverse.

Fig. 4.

$$\begin{array}{r} 4 \\ \times\ 4 \\ \hline \end{array}$$

Reverse.

The side represented by Fig. 1 was the side used for drill. The side represented by Fig. 2 was that used for testing in addition. A similar arrangement was used for multiplication. Figures 3 and 4 above indicate the disposition of the numbers having no reverse form. The reason for the different arrangement of the numerals on the obverse and the reverse sides arises out of practical considerations. Figures 2 and 4 show the arrangement that is usually presented when problems are to be solved, while the arrangement shown in Figures 1 and 3 recommends itself for convenience in printing and in writing results.

The teachers being already expert in the use of perception cards in the teaching of reading became quite as expert in a very short time in their use for teaching the combinations of number. These cards proved to be important savers of the teachers' time and effort as well as effective means of visual instruction.

4. THE TIME AND METHOD OF TESTING.

At the beginning of the number period each morning a test involving the facts of the group drilled upon the day before was given; hence, there was always at least one day's interval between the drill and the test upon the group under study. Frequently, of course, a much longer time elapsed, as when a test came after a single holiday, when there was a two-day interval, or when it came on Monday, thus giving a three-day interval, &c. For the tests the teacher either wrote the combinations on the blackboard, or, which was the usual method, arranged the perception cards containing the proper combinations in the chalk tray of the blackboard. The pupils copied these in the order of their arrangement and wrote under each the answer. These papers were then collected and the results recorded by the teacher.

a. Objections.—It is recognized that two objections may be offered to the written test as a means of gathering data of this kind, especially with first grade children: (1) the difficulty that is experienced by them on the mere mechanical side of writing the symbols is likely to detract from accuracy in the process involved; and (2) an increased opportunity to obtain answers by

"counting up" in the case of addition, or "running down" the tables in multiplication. On the other hand, it was a matter of observation that children learned very quickly to express themselves by means of the written symbol. For the most part they had come into the first grade from a kindergarten and reception grade where they had been taught to count, to recognize numbers, and to make the symbols on the blackboard. They had also had some practice in the use of the pencil for this purpose. As to "counting up" and "running down" the rows to reach the desired point, everything in the method of presentation and the teacher's attitude was made to discourage those devices.

b. Advantages.—The written method has the positive advantage of being free from the excitement or the stimulus of personal contact with the questioner. It may be said to represent the calm judgment of the individual tested, just as the written productions of mature persons represent a better quality of thought and expression than their unprepared oral efforts. But the greatest advantage of this method was its adaptability to the purposes which it served. It would have been impossible without serious interference with the other work of the school for the teacher to have given an oral examination to each of her pupils in each of the facts of each group apart from the other members of the class each day of the week.

5. RECORDING RESULTS.

For recording results a special blank was prepared. This blank contained the entire record of a class throughout its work on a given group, from the preliminary test to the "review" test, two weeks after the last intensive drill. Since positive results from the standpoint of teaching were desired, a standard of at least 97% class efficiency being the aim in each group, the totals and percentages were based upon the correct answers in the various tests. As the results were all to be subsumed under the forty-five addition and the seventy-eight multiplication combinations, an error in either the direct or the reverse form or both was counted simply as one error, and a "×" was marked in the proper place to indicate the same. Correct results were indicated

by a check mark (√). The percentages each day were based upon the number of pupils present. By this method the teacher had before her each day the standing of the individuals, the standing of the combinations themselves, and, in the summary, the class accomplishment. It is interesting to note here that after the tests were all completed the teachers had found the scheme so helpful that many of them requested that they might be allowed to continue the use of this record.

6. NEW GROUP TESTS.

When all of the groups of the combinations had been taught and the "review" test on the last one had been given, the addition combinations were divided into three groups, and a test was given on each of these new groups. After each of such tests was given, the five combinations in which errors were most frequent were again taken up for intense drill and dealt with in the same manner as originally, under the heading "second time over" in the final summary of results If frequent errors appeared in more than five combinations, then the other combinations in which errors appeared were also given intensive drill, and so on until the difficulties had been covered a second time, unless the close of the term prevented, as was the case in a few instances.

The multiplication combinations were similarly divided, but into five groups. These groups both in addition and multiplication contained for the most part both the direct and the reverse forms of the number facts. Here again the combinations showing most errors were again taken up and drilled intensively.

Unfortunately the results of those tests were not individualized, that is, were not placed in the individual records of the pupils taking them. Consequently they could not be included in the general summary of results because of the fact that a great number of the pupils had to be eliminated from the results in various groups owing to absence. The records of the "second time over" drills would seem to indicate that most of the errors made in these tests must have been made by the pupils thus eliminated.

7. FINAL TESTS.

At the end of the term final tests in the forty-five addition and the seventy-eight multiplication facts were given with the same kind of test sheets as were used in the preliminary tests at the beginning of the term (See IX, 1). The results of these tests were tabulated for each pupil and appear for the pupils not eliminated in the final summary sheets (See Table III) under the head of "Final Tests."

X. PSYCHOLOGICAL JUSTIFICATION OF THE METHOD OF PRESENTING THE COMBINATIONS.

1. THE NUMBER CONCEPT.

To name things is an inherent tendency of the human mind. To represent these names by means of symbols is the next step in the natural course of its development—a step that may have required many centuries to take. No conscious reflection, no analysis, no synthesis is necessary to the process of naming. A child calls water "maw," as the writer once knew a child to do. Aside from the conventions of language, his name for this substance was quite as good, and did serve the purposes of his limited environment quite as well, as the word in common use. The symbol, by which we mean the written symbol, is the product of reflection, of analysis, of synthesis. The symbol must of necessity have some relation, either real or fancied, either to the object or to its name. Numbering differs from naming in that it arises not from a perceptive process, which is of one thing or more than one as far as numbering goes, but, like the process of developing symbols, it arises out of reflection about things. Presumably people who have small powers of reflection, such as the savage races, will have very few number concepts. And that this is the case has already been pointed out by Branford (1). According to McLellan and Dewey, "Number is not (psychologically) got *from* things, it is put *into* them" (16). Nor yet is it gotten with-

out things. So that to arrive at number concepts without the aid of things would be as impossible as to arrive at the concept "Greek Education" without the aid of symbols.

2. PUTTING CONTENT INTO NUMBER AND PROCESS SYMBOLS.

Content is put into symbols of number by showing their relation to the concrete things for which the symbols stand, whether it be by measuring (1), or by counting (24), or by ratio (34), it matters little, adequate number concepts having been obtained in all of these ways. Content is put into symbols of process by showing with things the relations indicated by such symbols. To make clear the content of number and process symbols by the use of objects was the object of the two objective presentations of each combination within the limits set forth.

3. OBJECTIVE DEMONSTRATION OF NUMBER FACTS AND PROCESSES.

The number of objective presentations was limited to two. This was done to avoid the possibility of forming a habit of depending upon objects and using the methods of counting to obtain results. It was thought that this number of objective presentations would be sufficient to demonstrate the meaning and purpose of the process involved. On this point Dr. Smith has the following to say: "It is important to use objects freely wherever they assist in understanding number relations, but it is equally important to abandon them as soon as they have served their purpose. * * * To continue to use objects after they have ceased to be necessary is like always encouraging a child to ride in a baby carriage" (32).

The number of objects used in teaching addition facts was restricted to ten for the reason that more than that number tends to confuse and encumber the process rather than to clarify it, while the pupil not being able to visualize the larger groups is forced into the very method which we strive to avoid—the counting by ones (42).

4. MEMORY AND HABIT.

The facts of a given group when once developed make some impression on the mind of the child, but that impression is insufficient for the purpose of instruction, which is, that each fact should become a permanent and instantly available part of his mental equipment. The reaction to the abstract stimulus $3 + 4$, or $\begin{array}{r} 3 \\ +\,4 \\ \hline \end{array}$ should be "7" without the slightest hesitation, just as the reaction to m-a-n when seen or spoken is at once "man." To obtain this result the laws of memory and habit must be invoked, for the correct reactions must not only be memorized but they must be made habitual. The most important mechanical factor in the production of both these results is *repetition* (11). There must be a great deal of repetition of the combinations in the abstract, repetition freed from objective paraphernalia in the usual sense, but repetition which may make use of various number devices which lend interest to the exercise. We may be pardoned for again quoting Dr. Smith. "It is a serious error," he says, "to neglect abstract drill work in arithmetic. So far as scientific investigations have shown, pupils who have been trained chiefly in concrete problems to the exclusion of the abstract work are not so well prepared as those in whose training these two phases of arithmetic are fairly balanced. . . . At the same time it (abstract computation) is the most practical part of arithmetic, since most of the numerical problems we meet in life are simplicity itself as far as the reasoning goes; they offer difficulties only in the mechanical calculations involved, and constantly suggest to us our slowness and inaccuracy in the abstract work of adding, multiplying and the like. In the first grade this work is largely but not wholly oral" (33).

5. ASSOCIATION, JUDGMENT, ETC.

Memory for facts can, of course, be aided by richness of association, by judgment, by emotional appeal, and by voluntary attention. While all of these may and should play some part in

memorizing number combinations, the opportunities for their use for this purpose as compared with similar opportunities presented by poetry, or facts of history or geography, &c., are few. On this point it is interesting to note Dr. McMurry's observations. He says that "advantage should be taken of the similarity between the 2's, 4's, and 8's, also between the 3's, 6's, and 9's. Note also the fact that in counting by 8's the right-hand figure expresses two less each time, and in counting by 9's one less. The children are interested and curious about these things, and they aid the memory. The number 24 equals 4×6, 3×8, and 2×12. Such cases should be examined, and reasons given for the variety of factors in the same product.

"In learning the multiplication tables later, the memory can be aided by a variety of these reviews, comparisons and rational analyses.

"First of all, additions based upon objective work and measurement stand in the background of thought, as giving meaning to multiplications and divisions. Secondly, the repetitions and regularities running through some of the tables should be studied as of curious interest, as in the case of the 10's and 5's, 4's and 8's. Third, the identity of certain products in different tables, as $4 \times 5 = 20$, and $5 \times 4 = 20$, and $2 \times 10 = 20$. A comparison and analysis of these identities is excellent thought work and a positive aid to the memory" (18). All these things are true, but they alone cannot be relied upon to give the child that facility in the use of the number combinations which will make him adept in their promiscuous use, in which form only they will be required of him in their practical applications.[1]

It was noted in teaching the addition combinations that when a double number, as $4 + 4$, was mastered it formed a valuable connection with $4 + 3$, and $4 + 5$, the former being one less and the latter one more than the sum of the double number. This principle was also sometimes extended to sums two more or two less than the sum of the double number. Then, too, after drill

[1] The fact that $56 = 7 \times 8$, gives the natural number sequence of 5, 6, 7, 8 may be helpful, but the possibility of error is still large. The scheme for remembering the table of 9's is interesting and should be pointed out to the learner, but not relied upon to take the place of necessary drill.

there was established a sense of fitness of answer to combination. One day the writer was in a 1–B grade (lower half) when the teacher at his suggestion called one of her boys to the desk and said, "Herbert, what are 8 and 5?" There was a moment's hesitation, then came the answer "13." "What were you doing while we were waiting for the answer?" asked the writer. The reply was, "I was thinking over the numbers." This doubtless meant that he was going over the number series until he came to the one that "fit" $8 + 5$. Professor Thorndyke calls all such combinations "paired associates" (37). Given the stimulus 4×7, the neural discharge is into "28," just as the stimulus of the German word "Gedächtnis," for instance, discharges into "memory" for the English student of that language. For a great many of the combinations the possible associations are just as remote as the relation between *Gedächtnis* and *memory* to one who knows no German, while in others the relations are so intricate that to remember the relation is more difficult than to commit to memory the combination.

6. REPETITION AND RETENTION.

Dr. Yocum says, "The dominant factor of any method which has for its end maximum certainty and readiness of recall of the subject-matter to be taught is *repetition*" (47). Mr. Speer declares, "The way to succeed (in memorizing the tables, and he believes in taking only a few of the facts of each at a time) is to develop vivid mental pictures, and to fix these pictures by bringing them again and again before the mind" (35). Dr. W. T. Harris went so far as to say that, "Lists of names, * * * also numbers, as, *e. g.*, the multiplications, the melting points of minerals, &c., must be learned without aid. All indirect means only serve to harm here, and are required as self-discovered devices only in case that interest or attention has been weakened" (6). Dr. Rosenkranz himself declares (and the above opinion of Dr. Harris is a comment on his belief), "The means to be used (and these are based on the nature of memory itself) are, on the one hand, the pronouncing or writing the names or numbers, and, on the other, repetition; by the former we gain distinctness, by the latter sureness of memory" (7).

7. EXTENT OF THE APPEAL TO THE SENSES.

It will be noted that, while repetition is the dominant feature of the drill given, this repetition involved as many of the senses as possible. First, it was made to appeal to the eye in the case of objective development, then further to the sense of vision as symbols by use of number perception cards specially prepared; second, to the motor and auditory senses in the oral drill, then further to the visual and motor sense in the written repetitions. Finally, teachers were instructed in general talks to make use of the instincts of play and of rivalry to accomplish results.

8. SOME EXPERIMENTS SHOWING THE RESULTS OF VARYING MODES OF REPETITION IN THE PROCESS OF MEMORIZING.

In this connection it is interesting to note some recent experiments in committing to memory that have been summarized by Miss Elizabeth L. Wood of Clark University (43). Under the head of "Methods of Presentation and Learning" she concludes that the evident "want of agreement shown in the studies so far described" evinces the fact that the problem of whether an auditory, a visual, a visual-auditory, or a visual-auditory-motor presentation is "far from solution." Both Pohlman (25) and Meumann (20) believe "that pupils should be taught to use all sorts of presentations though not forced into uneconomical methods" (8). Von Sybel (40) in 1909 made experiments of similar import with 17 adults, students of rank; so also in 1912 Professor Henmon (9) of Colorado University reported experiments with six of his students from which he deduced the following interesting conclusions:

"I. Auditory presentation is best in almost all cases.

"II. Visual-auditory presentation is superior to visual alone in 87% of the cases.

"III. Visual-auditory-motor is inferior to visual-auditory in 59% of the cases; is inferior to auditory alone in 55% of the cases, to visual alone in only 15%. Therefore, a simultaneous appeal to several sense departments at once is of no advantage.

"IV. The relative value of the different modes of presentation remains unchanged for one, two, or three presentations.

"V. Individual differences in the amount retained was high."

Thorndyke (38), Kuhlman (12), Segal (30), and Weber (41) have also made interesting and important investigations along this line. Among these Kuhlman finds, in direct opposition to Henmon, that the poorest results come from auditory consciousness. Segal concluded that every individual should keep to his own type of imagery—rather a difficult pedagogical principle for a class room teacher with forty different individuals to teach to put into practice, yet doubtless of much significance after all, if every such teacher could be an expert psychologist and investigator and be able to divide her class into sections, one containing only *visuelles,* another *auditives,* another *motives,* &c., according to the individual memory proclivities of the pupils.

The results of Weber show that retention is in direct relation to the number of repetitions, other things being equal. This is also in general accord with experiments in committing memory gems, made by the writer some years ago, the results of which are on file in the Department of Education of the University of Pennsylvania (10). Another conclusion of Weber, also especially significant in our method of presenting this work, was that "it is economical to divide big tasks to be memorized into smaller ones, recognized as parts of the whole, but which can be learned without exhaustion."

In the light of these experiments, which unfortunately were carried on with groups too small to make their conclusions more than tentative, our method of presenting the combinations cannot be objected to on psychological grounds.

Lest anyone should misunderstand this statement, however, and hold that we have violated both pedagogically and psychologically the very principle of practice the necessity of which this study seeks to demonstrate in the very first sentence of our "Directions to Teachers for Conducting the Drills and Tests" (IX, 2), we must urge that the validity of our results depended upon the rigid adherence to the practice of equal daily emphasis, even though it did violate what we know perfectly well to be the best practice in teaching. Without this uniformity of em-

phasis, it would have been impossible to judge relative difficulty by the results obtained. If just the right emphasis had been put upon each combination for its mastery, we should have expected just the same number of errors to appear in 4×3 as in 8×6, if errors appeared at all, and relative difficulty would have been already worked out in practice, a situation which is contrary to fact. So much then had to be sacrificed for the sake of the experiment.

XI. DIFFICULTY.

1. ITS MEANING.

Since this is a study concerning difficulty, it would seem advisable to discuss, at least briefly, some of the general characteristics of the term. It would also seem advisable to discuss it in the light of the various phases of our investigation.

In teaching we impart knowledge; by stimulating continuous use of that knowledge we develop skill. The first three phases of our experiment, namely, the determination of relative difficulty as shown in learning the facts, then as shown by the number of errors made, and finally as shown by the comparative number of combinations forgotten, we are dealing for the most part with knowledge; in the last phase of our investigation, namely, the time tests, our principle interest is the measurement of skill. We are concerned, in the first place, with relative difficulty from the standpoint of acquisition and retention, and, in the second place, we note relative difficulty from the standpoint of use.

Difficulty itself is a relative term. Existing in a small degree it represents facility, or ease of accomplishment. In its highest degree it requires the addition of a modifying word to convey its meaning. Then, too, what is difficult for one individual may be very easy for another. What is difficult for a given individual at a given time or under certain circumstances may be very easy at another time or under other circumstances. Any experiment, therefore, which seeks to determine relative difficulty must clearly set forth, as we have sought to do in this, the details indicated by the following questions: difficult for whom, at what age and grade, and under what method of instruction?

2. ITS CAUSE.

As has already been pointed out, memory for facts is assisted by richness of association, use of judgment, &c. If there are combinations around which the associations are few or remote, as in the case of $8 + 5 = 13$, or $9 \times 7 = 63$, where the figures in the answer are all different from those in the combination, as Professor Döring notes (2); if their nature is such as to make difficult the use of judgment, as in the case of $8 + 6 = 14$, or $8 \times 12 = 96$, where the very bigness of the results is on or beyond the borderland of the comprehension of the pupil, we may expect difficulty in acquisition and in retention, and slowness and inaccuracy in use. If, on the other hand, the associations are many and easy, occurring in the child's daily activities, as in the value of coins, in the purchase of rolls from the baker, or bananas, or pieces of candy from the shop-keeper, in the playing of marbles or dominoes, &c.; if the answers contain numbers which occur in the combinations, as $7 \times 5 = 35$, or $8 \times 6 = 48$; if the judgments are easy, as $5 + 5 = 10$, then $5 + 6 = 11$; if the answers are well within the experience or easy comprehension of the child —in all these cases we may expect little difficulty either in acquisition or retention, and we may look for a high degree of skill and accuracy in the use of the combinations.

3. ITS MEASURE.

As the acquisition of knowledge means, psychologically, the formation of new "brain paths," as Professor James expresses it, difficulty might be expressed in terms of neural resistance—in mental "ohms," as it were. Aside from the physiological condition of increased blood supply to the brain during the process of special activity of that organ, we do not know enough of the nature of its functioning to measure that functioning in comprehensible terms. We must content ourselves at present by trying to measure it in terms of its external phonomena, as we measure that mysterious force called electricity. How do we determine that an

activity is difficult? In the first place, other things being equal, by the time it takes to learn to perform it with precision. Writing is a difficult art for the child to acquire. Months, perhaps years, of practice are necessary to develop freedom and skill in its use. In the next place, we determine difficulty by the persistence of error in the course of mastery. How many false notes are struck in the process of learning the scales; how many are made too long or too short, in learning to play a musical instrument? The smaller the chance for error, the less difficult the process. It is, therefore, less difficult to learn the piano than the pipe organ. One-step problems in arithmetic are less difficult than two or three-step problems.

We should, therefore, evaluate our data in the first experiment by correlating the time required to reach the results obtained with the number of errors made in the process. In attempting to do this we found that the time given to each of the groups was not a true index of difficulty, because, for the sake of *class* efficiency, the drills had to be prolonged beyond the time necessary to secure efficiency with those who were not eliminated on account of absence from some necessary part of the work. The girls had sometimes to "mark time" while the boys were catching up. In the last experiment, however, as time was the only element considered, only correct results being counted, we may be justified in using it as a true index of difficulty as far as a comparison of the groups of facts indicated is concerned.

The factor of time having, in a measure, been eliminated by force of circumstances in the evaluation of the results of the first experiment, we have laid down as fundamental the following two methods of evaluation: (1) In the process of learning the combinations, *the co-efficient of difficulty is the ratio of the number of errors made in the process to the number of errors overcome, as shown by a comparison of the number of errors made in the preliminary test with the number of errors made in the final test in each combination, and reduced, for the purpose of comparison, to a basis of 50 or 100 pupils.* In calculating co-efficients from this point of view, errors in the review tests were counted the same as errors in the course of instruction. (2) *The co-efficients of difficulty by the second method of evaluation are determined by dividing the*

grand total of errors made in each combination by the number of pupils involved. This method gives as the co-efficient of difficulty the average number of errors for each pupil. The method is based upon the following reasoning: the presence of error in the preliminary test is an indication of difficulty—the difficulty to be overcome; the presence of error in the process of instruction is an indication of difficulty—the difficulty of mastery; the presence of error in the review and in the final test is an indication of difficulty—the difficulty of retention; hence the total of these errors would seem to indicate the entire difficulty presented by the combination. We have considered both of these methods of such value as to be worthy of record and have given the results of each side by side in Table VI.

In the second and third experiments we have worked on the assumption that the number of errors made in the tests is in direct proportion to the difficulty, and have arranged the results in these experiments accordingly. Difficulty then is to be measured by the "method of errors" in the first three parts of our investigation, and by the "method of time" in the last part of the investigation.

XII. RESULTS IN THE STUDY OF ADDITION.

1. RESULTS IN TEACHING THE FORTY-FIVE ADDITION COMBINATIONS.

To give in full the tabulation for all the classes for all the days for all of the combinations would require more space than is at the disposition of the writer. Those who may be interested may have access to the complete records by applying to the Department of Education of the University of Pennsylvania. However, in order to show the method pursued, the data tabulated for the study of $8 + 3$ is given below as typical.

TABLE III.

BOYS.

COMBINATION, 8 + 3.	No. Pupils.	Grade.	Average Age.	Number of Errors. Preliminary Test.	1st Day.	2d Day.	3d Day.	4th Day.	5th Day.	6th Day.	7th Day.	8th Day.	9th Day.	10th Day.	Review Test.	Final Test.	Total Errors.	Grand Total Errors.	Errors Co-efficient.
	5	1B	6	4	0	1	0	0	0	0	....	....	....	....	0	1	6	....	.....
	2	1A	7	1	0	0	....	....	....	....	....	....	....	....	0	0	1	....	.....
	4	1A	7	2	2	0	0	1	0	0	0	....	....	....	1	0	6	....	.....
	7	2B	7½	4	0	0	....	....	....	....	....	....	....	....	0	1	5	....	.....
	12	1B	6	11	1	1	1	1	3	1	1	0	2	0	1	4	27	....	.....
	13	1A	6½	4	3	2	1	1	1	....	....	....	....	....	3	1	16	....	.....
	4	2B	7½	2	3	2	2	2	0	1	1	....	....	....	0	0	13	....	.....
	6	2B	7½	0	0	0	....	....	....	....	....	....	....	....	0	0	0	....	.....
Totals	53	....	....	28	9	6	4	5	4	2	2	0	2	0	5	7	74	....	.....
Errors second time over...	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	2	....	.....
																		76	1.43

GIRLS.

COMBINATION, 8 + 3.	No. Pupils.	Grade.	Average Age.	Preliminary Test.	1st Day.	2d Day.	3d Day.	4th Day.	5th Day.	6th Day.	7th Day.	8th Day.	9th Day.	10th Day.	Review Test.	Final Test.	Total Errors.	Grand Total Errors.	Errors Co-efficient.
	7	1B	6	5	0	0	0	1	0	0	....	....	....	....	0	0	6	....	.....
	7	1A	6½	3	0	1	....	....	....	....	....	....	....	....	0	0	4	....	.....
	5	1A	7	5	2	1	2	0	1	0	0	....	....	....	2	0	13	....	.....
	6	2B	7½	1	2	0	....	....	....	....	....	....	....	....	0	0	3	....	.....
	9	1B	6	9	3	4	0	2	2	1	0	1	0	0	0	5	27	....	.....
	8	1A	6½	1	0	0	0	0	0	....	....	....	....	....	0	1	2	....	.....
	6	2B	7½	0	1	0	0	0	0	0	0	....	....	....	1	0	2	....	.....
	8	2B	7½	5	0	0	....	....	....	....	....	....	....	....	0	1	6	....	.....
Totals	56	....	....	29	8	6	2	3	3	1	0	1	0	0	3	7	63	....	.....
Errors second time over...	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	0	....	.....
																		63	1.13
Grand Totals.......	109	....	....	57	17	12	6	8	7	3	2	1	2	0	8	14	....	139	1.28

The same kind of a study was made of each of the forty-five addition facts, an entire summary of which, tabulated to show co-efficients of difficulty from the two points of view as set forth on page 38, is shown below:

TABLE IV.—PART I.

SUMMARY OF RESULTS IN TEACHING ADDITION.

	Boys.								*Girls.*							
	a	b	c	d	e	f	g	h	a^1	b^1	c^1	d^1	e^1	f^1	g^1	h^1
1.......	57	12	41	1	11	3.73	3.27	0.95	48	6	34	2	4	8.50	8.85	0.88
2.......	57	28	59	5	23	2.56	2.25	1.62	48	18	45	4	14	3.20	3.33	1.40
3.......	57	29	63	2	27	2.33	2.04	1.65	48	21	39	2	19	2.05	2.14	1.29
4.......	57	32	55	6	26	2.12	1.86	1.63	48	19	41	3	16	2.56	2.66	1.31
5+1....	57	29	59	3	26	2.27	1.99	1.60	48	19	43	3	16	2.69	2.80	1.35
6.......	58	25	57	2	23	2.48	2.10	1.45	55	22	22	4	18	1.22	1.11	0.87
7.......	58	27	52	2	25	2.08	1.79	1.40	55	19	17	5	14	1.21	1.10	0.75
8.......	58	26	49	2	24	2.04	1.76	1.33	55	23	19	3	20	0.95	0.86	0.82
9.......	58	29	57	3	26	2.19	1.89	1.53	55	23	23	5	18	1.28	1.16	0.93
2.......	58	22	46	2	20	2.30	1.98	1.22	55	20	40	3	17	2.35	2.13	1.15
3.......	61	28	64	4	24	2.66	2.18	1.57	63	31	45	3	28	1.61	1.28	1.25
4.......	61	32	57	5	27	2.11	1.73	1.54	63	30	51	4	26	1.96	1.55	1.35
5.......	61	23	66	3	20	3.30	2.70	1.51	63	28	54	4	24	2.25	1.78	1.36
6+2....	61	21	68	4	17	4.00	3.27	1.53	63	34	47	5	29	1.62	1.28	1.36
7.......	61	38	71	3	35	2.03	1.66	1.84	63	38	49	11	27	1.81	1.43	1.55
8.......	57	25	37	3	22	1.68	1.47	1.14	54	15	24	5	10	2.40	2.22	0.86
9.......	57	28	42	7	21	2.00	1.75	1.35	54	24	26	3	21	1.24	1.15	0.98
3.......	57	25	29	4	21	1.38	1.21	1.02	54	23	18	5	18	1.00	0.93	0.85
4.......	57	25	38	8	17	2.24	1.96	1.25	54	20	44	6	14	3.14	2.91	1.30
5.......	57	28	36	6	22	1.63	1.43	1.23	54	22	48	8	14	3.43	3.18	1.45
6+3....	53	20	36	6	14	2.57	2.42	1.17	56	32	32	9	23	1.39	1.24	1.30
7.......	53	29	32	4	25	1.28	1.21	1.23	56	27	33	8	19	1.74	1.55	1.21
8.......	53	28	41	7	21	1.95	1.84	1.43	56	29	27	7	22	1.23	1.10	1.13
9.......	53	18	43	5	13	3.31	3.12	1.25	56	31	38	7	24	1.58	1.41	1.36
4.......	53	12	16	3	9	1.78	1.68	0.59	56	20	28	2	18	1.55	1.38	0.89
5.......	55	27	34	4	23	1.48	1.35	1.18	55	25	43	6	19	2.26	2.06	1.35
6+4....	55	30	51	3	27	1.89	1.72	1.53	55	24	41	10	14	2.92	2.55	1.36
7.......	55	29	41	7	22	1.86	1.69	1.40	55	23	39	5	18	2.16	1.96	1.22
8.......	55	29	44	9	20	2.20	2.00	1.49	55	25	35	9	16	2.18	1.98	1.25
9.......	55	33	50	9	24	2.08	1.89	1.67	55	34	42	8	26	1.61	1.46	1.53
5.......	53	18	10	2	16	0.62	0.58	0.57	60	16	14	3	13	1.08	0.90	0.55
6.......	53	24	25	6	18	1.39	1.31	1.04	60	32	26	9	23	1.13	0.94	1.12
7+5....	53	29	36	7	22	1.63	1.53	1.36	60	35	35	11	24	1.46	1.21	1.35
8.......	53	22	33	9	13	2.54	2.39	1.21	60	36	36	11	25	1.44	1.20	1.38
9.......	53	27	26	11	16	1.62	1.53	1.21	60	37	31	8	29	1.07	0.89	1.27

TABLE IV.—PART I.—*Continued.*

SUMMARY OF RESULTS IN TEACHING ADDITION.

	Boys.								*Girls.*							
	a	b	c	d	e	f	g	h	a^1	b^1	c^1	d^1	e^1	f^1	g^1	h^1
6.......	54	21	18	3	18	1.00	0.92	0.79	45	20	19	5	15	1.27	1.41	0.98
7.......	54	21	49	8	13	3.77	3.49	1.45	45	27	28	8	19	1.47	1.63	1.40
8+6....	54	33	55	12	21	2.62	2.42	1.85	45	24	35	10	14	2.50	2.77	1.53
9.......	54	37	66	9	28	2.36	2.18	2.04	45	28	35	9	19	1.84	2.04	1.60
7.......	54	30	26	6	24	1.08	1.00	1.15	45	22	23	4	18	1.28	1.42	1.09
8+7....	66	47	76	12	35	2.17	1.64	2.05	57	42	67	8	34	1.97	1.70	2.05
9.......	66	42	96	12	30	3.20	2.42	2.27	57	45	75	13	32	2.34	2.10	2.33
8.......	66	43	47	7	36	1.31	0.99	1.47	57	36	27	7	29	0.93	0.81	1.23
9+8....	66	43	85	15	28	3.04	2.30	2.17	57	43	53	9	34	1.56	1.37	1.84
9+9....	66	42	47	7	35	1.34	1.01	1.45	57	38	35	3	35	1.00	0.88	1.33

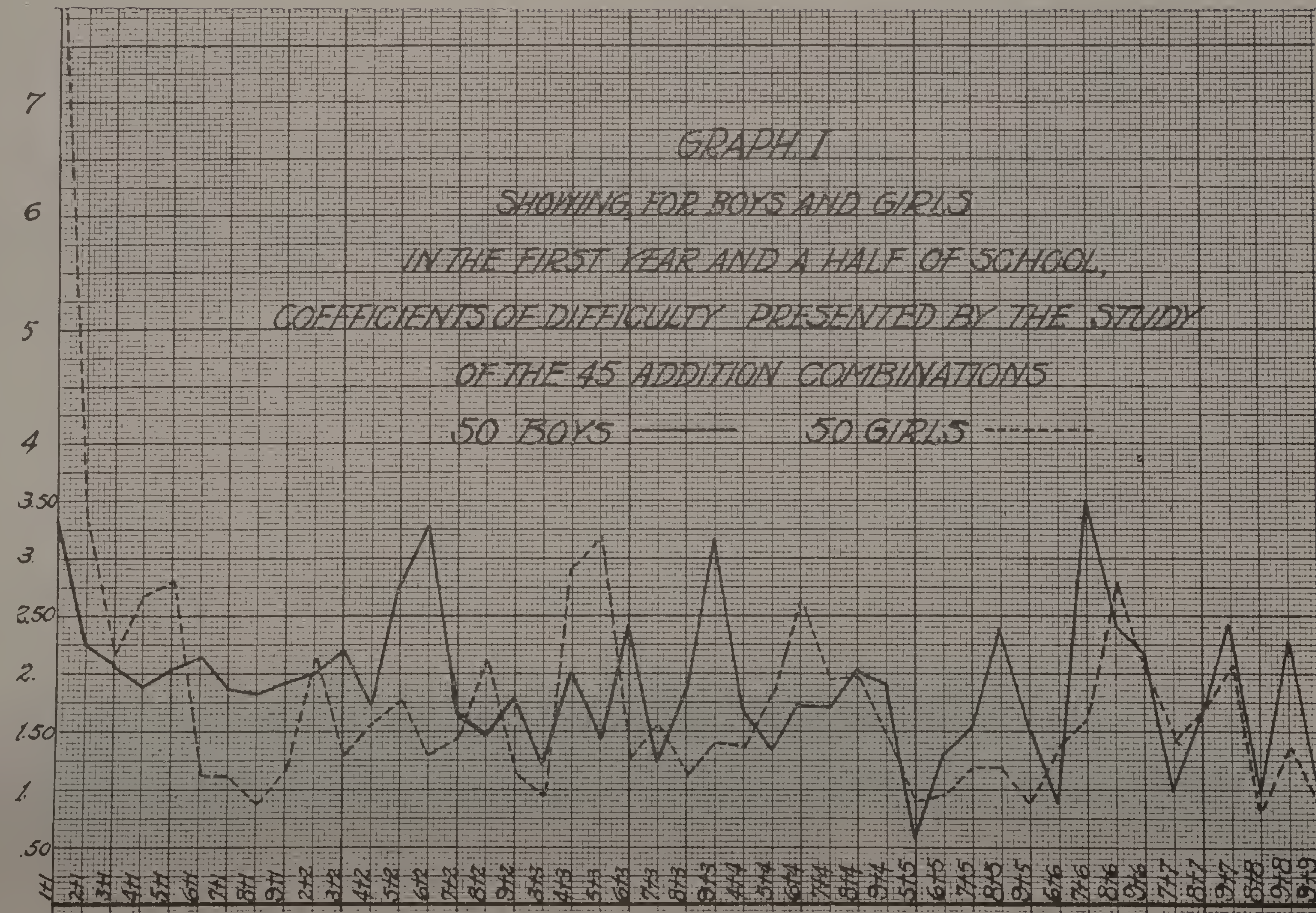
NOTE.—a and a^1, number of boys and girls, respectively; b and b^1, number of errors made by boys and girls, respectively, in the preliminary test; c and c^1, number of errors made in learning; d and d^1, number of errors in final test; e and e^1, (b — d) number of errors overcome; f and f^1, co-efficients of difficulty (c ÷ e); g and g^1, co-efficients of difficulty for 50 boys or 50 girls (50f ÷ a); h and h^1, total errors co-efficients of difficulty (b + c + d) ÷ a.

TABLE IV.—PART II.

TOTALS FOR BOYS AND GIRLS.

	A	B	C	D	E	F	G	H
1	105	18	75	3	15	5.00	4.76	0.91
2	105	46	104	9	37	2.81	2.68	1.51
3	105	50	102	4	46	2.22	2.11	1.49
4	105	51	96	9	42	2.28	2.17	1.48
5+1	105	48	102	6	42	2.43	2.31	1.56
6	113	47	79	6	41	1.68	1.49	1.17
7	113	46	69	7	39	1.77	1.56	1.08
8	113	49	68	5	44	1.55	1.37	1.08
9	113	52	80	8	44	1.82	1.61	1.24
2	113	42	86	5	37	2.32	2.05	1.18
3	124	59	105	7	52	2.02	1.63	1.41
4	124	62	108	9	63	2.04	1.65	1.43
5+2	124	51	120	7	44	2.53	2.04	1.43
6	124	55	115	9	46	2.50	2.02	1.44
7	124	76	120	14	62	1.94	1.56	1.69
8	111	40	61	8	32	1.90	1.71	0.98
9	111	52	68	10	42	1.62	1.46	1.17
3	111	48	47	9	39	1.20	1.08	0.94
4	111	45	82	14	31	2.64	2.38	1.27
5	111	50	84	14	36	2.33	2.10	1.33
6+3	109	52	68	15	37	1.84	1.69	1.24
7	109	56	65	12	44	1.48	1.36	1.22
8	109	57	68	14	43	1.58	1.45	1.28
9	109	49	81	12	37	2.19	2.01	1.30
4	109	32	44	5	27	1.63	1.50	0.74
5	110	52	77	10	42	1.83	1.61	1.26
6+4	110	54	92	13	41	2.24	2.04	1.45
7	110	52	80	12	40	2.00	1.82	1.31
8	110	54	79	18	36	2.19	1.99	1.37
9	110	67	92	17	50	1.84	1.67	1.60
5	113	34	24	5	29	0.83	0.73	0.56
6	113	56	51	15	41	1.24	1.10	1.08
7+5	113	64	71	18	46	1.54	1.36	1.35
8	113	58	69	20	38	1.81	1.60	1.30
9	113	64	57	19	45	1.27	1.12	1.24
6	99	41	37	8	33	1.12	1.13	0.87
7	99	48	77	16	32	2.40	2.42	1.42
8+6	99	57	90	22	35	2.57	2.60	1.71
9	99	65	101	18	47	2.15	2.17	1.86
7	99	52	49	10	42	1.17	1.18	1.12
8+7	123	89	143	28	69	2.07	1.68	2.05
9	123	87	171	25	62	2.76	2.24	2.30
8+8	123	79	74	14	65	1.14	0.93	1.36
9	123	86	138	24	62	2.22	1.80	2.01
9+9	123	80	82	10	70	1.17	0.95	1.40

NOTE.—For explanation see note at bottom of Table IV.—Part I.


GRAPH I
SHOWING, FOR BOYS AND GIRLS
IN THE FIRST YEAR AND A HALF OF SCHOOL,
COEFFICIENTS OF DIFFICULTY PRESENTED BY THE STUDY
OF THE 45 ADDITION COMBINATIONS
50 BOYS ——— 50 GIRLS - - - - -
7
6
5
4
3.50
3.
2.50
2.
1.50
1.
.50
1+1
2+1
3+1
4+1
5+1
6+1
7+1
8+1
9+1
2+2
3+2
4+2
5+2
6+2
7+2
8+2
9+2
3+3
4+3
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6+5
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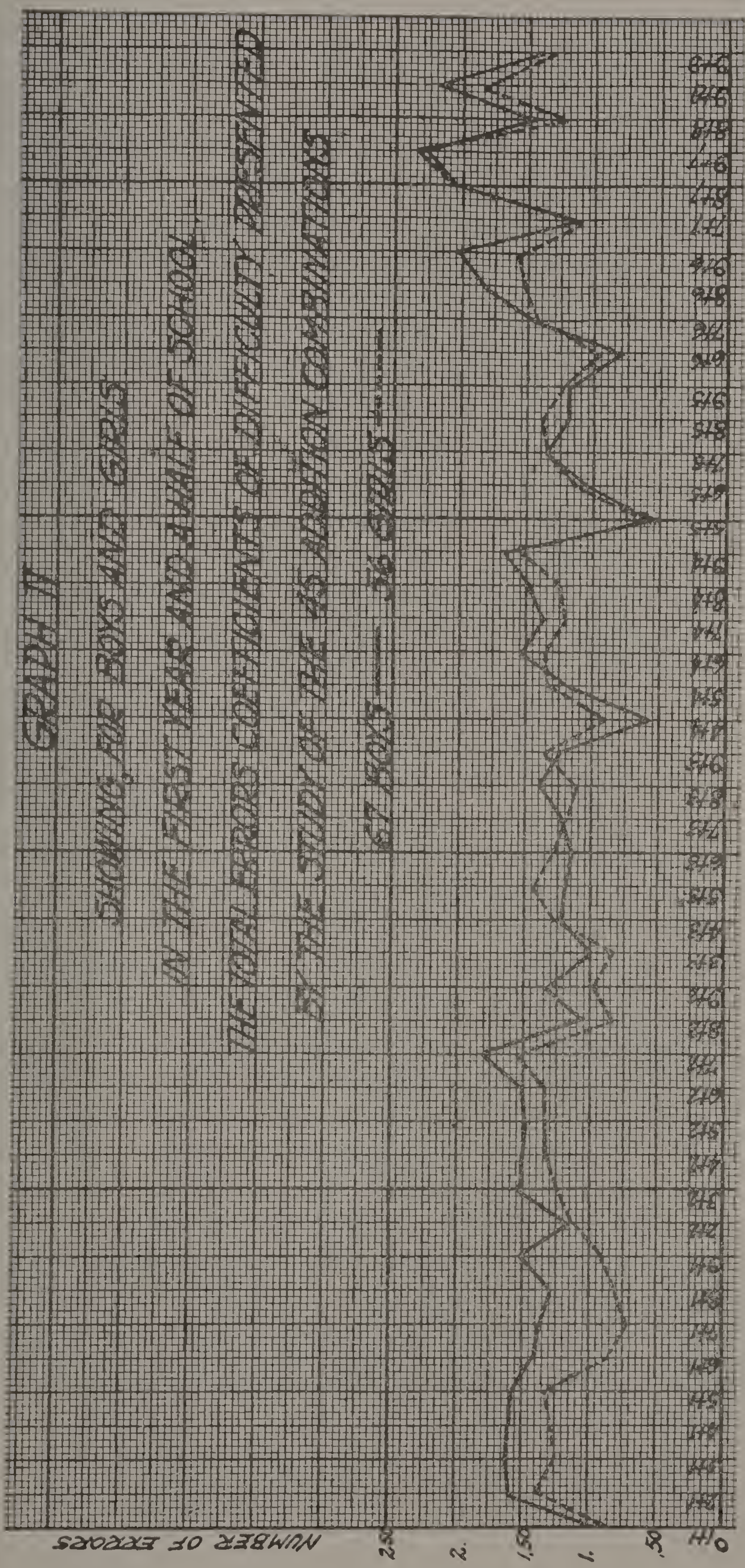
8+8
9+8
9+9


GRAPH II
SHOWING FOR BOYS AND GIRLS
IN THE FIRST YEAR AND A HALF OF SCHOOL
THE TOTAL ERRORS COEFFICIENTS OF DIFFICULTY PRESENTED
BY THE STUDY OF THE 45 ADDITION COMBINATIONS
67 BOYS —— 56 GIRLS ----
NUMBER OF ERRORS
2.50
2.
1.50
1.
.50
0

5. DISCUSSION OF RESULTS.

a. Interpretation of Table IV.—In the summary of Table III, shown in this table, it will be noted that the number of pupils varies for nearly every new group of facts presented. This is due to the elimination of pupils on account of absence. In order to be counted in the work a pupil must have taken the preliminary test, the review test, and the final test at the end of the term. Besides this he must not have been absent more than one day in every five days of intensive drill. As a result of these requirements at least 25% of the pupils who took part in the experiment had to be eliminated at one time or another in making up the results for the various groups.

Owing to this varying number it became necessary to reduce results to a common basis to facilitate comparison. It would be all right to compare with each other the co-efficients of difficulty obtained for boys for the first five combinations, for in each case there were 57 boys involved, but to compare the same with group three where there are 61 boys is unfair, as is shown by the following figures: the co-efficient of $4 + 1$ in column "f" is 2.12 for 57 boys, that of $4 + 2$ in the same column is 2.11 for 61 boys. A comparison here would indicate a difference of only .01, but when these figures are to be reduced to a basis of 50 boys each, as shown in column "g," the first co-efficient becomes 1.86 and the second 1.73, a difference of .13. The values in columns "f," "f^1," and "F" were at first reduced to a basis of one pupil. This, however, gave such a very small quantity that it was thought best to make the common basis a round number near in value to average number of pupils involved, so 50 was selected for the multiple of "f" and "f^1," and this required the multiple of "F" to be 100.

In this table the co-efficients shown in columns "g," "g^1," and "G" are those used for comparison for boys, girls, and the average for both boys and girls, while columns "h," "h^1," and "H" show in the same manner what we prefer to call the "Total Errors Co-efficient" for boys, girls, and the average for both boys and girls respectively.

b. Method of Studying the Graphs.—The values of the columns headed "g" and "g^1" (Table IV) are shown by the ordinate values of Graph I. In the study of the graphs in this part of the experiment we must always bear in mind that the co-efficients of difficulty found are co-efficients of difficulty *in the process of learning the combinations in the order of their presentation.* Let us take the three facts, 5 + 1, 4 + 3, and 8 + 4 for boys. It will appear that for each of these the co-efficient of difficulty is 2. We cannot reason from that that if these three facts were brought together they could be taught *de novo* with the same ease, but it does mean that the 5 + 1 fact at the time of its presentation offered the same degree of difficulty as the 4 + 3 and the 8 + 4 facts did at the time they were presented. If a line were drawn through the average heights of the ordinates in Graph I from left to right, its trend would be generally downward instead of upward as one might suppose. This demonstrates statistically in the study of numbers what everybody, even those who no longer believe in the doctrine of formal discipline, recognizes to be true, viz., that one of the results, one of the bi-products, of accomplishment in any field of endeavor is *power*—power to overcome with greater ease other difficulties in the same or in related fields of endeavor.

c. Analysis and Significance of the Graphs.—A further analysis of Graph I shows that in 14 out of the 45 cases the "curves" for boys and girls tend in opposite directions as in the case of 5 + 3, for instance. In one case, 8 + 5, the "curve" for girls is horizontal while for boys it tends strongly upward. In 30 cases, therefore, while the values for boys and girls differ materially, the tendencies are in the same direction as compared with the co-efficients of the combinations immediately preceding. The average of the co-efficients for boys is 1.89 while that for the girls is 1.82. This would indicate that the girls learned the combinations with slightly less difficulty than the boys, and this notwithstanding their inordinately large co-efficient for learning 1 + 1. Had the co-efficient for 1 + 1 been the same for both boys and girls, the difference between the averages of their respective co-efficients of difficulty would have been .18 instead of .07, or would

have shown that girls learn the combinations with about 10% less difficulty than boys of the same age and grade.

The high co-efficient of difficulty for 1 + 1 for the girls is due to the fact that only six errors were made by the 48 girls in this combination in the preliminary tests, the possibilities for improvement were, therefore, correspondingly less. If no errors, instead of two, had occurred in the final test, the co-efficient would then have been 5.91. Both of these figures demonstrate the necessity of studying the co-efficients here represented in connection with those shown in Graph II.

A glance at Graph II, which shows the total errors co-efficients of difficulty, calculated on the basis of the entire number of errors made by boys and girls in all of the tests, shows that the girls have a smaller average co-efficient than the boys, the average in this case being 1.41 for boys and 1.26 for girls, or about 10% less. This graph shows a very close similarity between the "curve" for boys and that for girls, the fewer errors appearing in the double numbers being especially significant for both.

We have no doubt that a part of the difficulty represented in the first group was due to the mechanical difficulty of writing the numbers required by the tests, though, as has already been explained, most of the 1–B children had had the writing of numbers in their work in the reception grade. This difficulty, however, would be at its maximum in September and applies to only about one-fourth of the pupils whose records are given.

In order to obtain the best results in the study of Graphs I and II, it is best to study them by groups of five combinations each, that is, in the manner in which the work was given. It is readily seen that the learning of the first group (1 + 1 to 1 + 5) carried over to make the second group (6 + 1 to 9 + 1) easier; but the fact 2 + 2 of this group represented a new order of things, consequently a new difficulty. With these graphs before her and an occasional check upon the results of her teaching, the primary teacher should be able to attack with some confidence the teaching of the elementary combinations in addition.

XIII. RESULTS IN THE STUDY OF MULTIPLICATION.

1. RESULTS IN TEACHING THE SEVENTY-EIGHT MULTIPLICATION COMBINATIONS.

For the sake of brevity, as in the case of the results of the study of addition, we give the detail of but one (8×7) of the seventy-eight combinations studied. This will serve to demonstrate the method of tabulating the results for all. The complete record may be had by applying to the Department of Education of the University of Pennsylvania.

TABLE V.

BOYS.

COMBINATION. 8 × 7.	No. Pupils.	Grade.	Average Age.	Number of Errors.										Total Errors.	Grand Total Errors.	Errors Co-efficient.
				Preliminary Test.	1st Day.	2d Day.	3d Day.	4th Day.	5th Day.	6th Day.	7th Day.	Review Test.	Final Test.			
	5	2A	8	5	2	1	1	2	0	....	....	1	2	14	.....	......
	15	2A	8	15	3	2	2	....	....	....	....	11	8	41	.....	......
	10	2A	8	7	1	1	0	0	0	....	....	1	7	17	.....	......
	11	3B	9	9	6	4	5	0	0	....	....	1	3	28	.....	......
	4	3B	9	4	0	2	2	0	1	0	....	0	1	10	.....	......
	14	3A	9½	8	2	2	1	1	....	....	....	2	1	17	.....	......
	18	3A	9½	15	2	1	1	....	....	....	....	10	7	36	.....	......
	3	3B	9	3	0	....	....	....	....	....	....	0	0	3	.....	......
	18	3A	9½	15	6	4	....	....	....	....	....	6	8	39	.....	......
Totals	98	.....	.....	81	22	17	12	3	1	0	....	32	37	205	.....	......
Errors second time over........	.....	.....	.....	.....	....	....	....	....	....	....	....	.....	.....	35	.....	......
															240	2.45
GIRLS.																
	4	2A	8	4	1	1	0	0	0	....	....	0	2	8	.....	......
	16	2A	8	16	5	1	1	....	....	....	....	9	2	34	.....	......
	9	2A	8	9	3	1	1	2	0	....	....	1	3	20	.....	......
	4	3B	9	4	3	0	0	1	0	0	....	0	2	10	.....	......
	8	3A	9½	3	1	3	1	0	....	....	....	1	1	10	.....	......
	17	3B	9	12	4	1	0	....	....	....	....	5	3	25	.....	......
	3	3B	9	2	0	....	....	....	....	....	....	0	2	4	.....	......
	13	3A	9½	11	2	0	....	....	....	....	....	6	1	20	.....	......
Totals	74	.....	.....	61	19	7	3	3	0	0	....	22	16	131	.....	......
Errors second time over........	.....	.....	.....	.....	....	....	....	....	....	....	....	.....	.....	28	.....	......
															159	2.15
Grand Totals	172	.....	.....	142	41	24	15	6	1	0	....	54	53	.....	399	2.32

From the full records, of which Table V is a sample, the data as shown in Table VI below were obtained. This table represents the tabulation of nearly 13,000 errors, and is worked out on exactly the same principles as Table IV in addition, and, therefore, needs no special comment as to its meaning.

TABLE VI.—PART I.

SUMMARY OF RESULTS IN TEACHING MULTIPLICATION.

	Boys.								*Girls.*							
	a	b	c	d	e	f	g	h	a^1	b^1	c^1	d^1	e^1	f^1	g^1	h^1
3*......	81	21	8	12	9	0.89	1.10	0.51	60	18	8	4	14	0.57	0.95	0.50
6.......	108	46	55	20	26	2.12	1.96	1.12	75	29	30	8	21	1.43	1.90	0.89
8×3....	108	45	73	29	16	4.56	4.22	1.36	75	39	34	10	29	1.52	2.02	1.24
9.......	97	35	54	25	10	5.40	5.56	1.18	70	30	48	9	21	2.33	3.33	1.24
11......	97	28	23	7	21	1.08	1.12	0.60	70	22	19	5	17	1.12	1.60	1.66
12......	97	32	50	21	11	4.55	4.69	1.06	70	31	37	8	23	1.61	2.30	1.09
4.......	97	47	23	18	29	0.79	0.81	0.91	70	34	19	4	30	0.63	0.90	0.77
7.......	95	49	91	26	23	3.96	4.16	1.75	64	40	46	7	33	1.39	2.17	1.45
8.......	95	52	126	28	24	5.25	5.52	2.17	64	33	58	9	24	2.42	3.78	1.56
9×4....	95	62	130	38	24	5.42	5.70	2.42	64	41	67	11	30	2.23	3.48	1.86
10......	99	29	15	5	24	0.62	0.63	0.50	76	30	11	4	26	0.42	0.50	0.59
12......	99	55	54	27	28	1.93	1.95	1.37	76	46	37	10	36	1.03	1.35	1.22
5.......	99	33	14	10	23	0.61	0.62	0.58	76	38	14	4	34	0.41	0.54	0.74
7×5....	101	46	51	28	18	2.83	2.80	1.24	72	48	31	7	41	0.76	1.05	1.19
10......	101	34	14	3	31	0.45	0.45	0.50	72	33	14	1	32	0.44	0.61	0.67
12......	95	60	31	23	37	0.84	0.88	1.20	77	46	19	9	37	0.51	0.61	0.96
6.......	95	50	30	13	37	0.81	0.85	0.98	77	45	19	20	25	0.76	0.99	0.92
7.......	95	68	101	23	45	2.47	2.60	2.13	77	55	69	12	43	1.60	2.08	1.77
8.......	95	66	98	39	27	3.63	3.82	2.14	77	60	75	14	46	1.63	2.12	1.93
9×6....	95	77	111	43	34	3.26	3.43	2.44	77	57	94	20	37	2.54	3.30	2.22
10......	98	43	10	8	35	0.29	0.30	0.62	74	38	4	3	35	0.11	0.15	0.47
11......	98	54	23	6	48	0.48	0.49	0.85	74	42	13	4	38	0.34	0.46	0.80
12......	98	72	67	25	47	1.43	1.46	1.67	74	58	46	10	48	0.96	1.30	1.54
7.......	98	74	37	14	60	0.62	0.63	1.28	74	47	21	9	38	0.55	0.74	1.09
8.......	98	81	112	37	44	2.77	2.82	2.45	74	61	82	16	45	1.82	2.46	2.15
9.......	88	84	184	43	41	4.49	5.10	3.53	69	61	76	21	40	1.90	2.75	2.29
10×7...	88	47	16	5	42	0.38	0.43	0.77	69	38	8	1	37	0.22	0.32	0.68
11......	88	52	31	8	44	0.71	0.80	1.03	69	36	12	4	32	0.37	0.54	0.75
12......	88	65	102	33	32	3.19	3.62	2.28	69	54	55	10	14	1.25	1.81	1.72

* Thirty-four of the smaller combinations have been omitted in this table for typographical reasons, but their several co-efficients are included in the graphs involved.

TABLE VI.—PART I.—*Continued.*

SUMMARY OF RESULTS IN TEACHING MULTIPLICATION.

	Boys.								Girls.							
	a	b	c	d	e	f	g	h	a^1	b^1	c^1	d^1	e^1	f^1	g^1	h^1
8.......	88	63	110	25	38	2.90	3.29	3.25	69	48	37	10	38	0.97	1.41	1.38
9.......	102	87	94	33	54	1.74	1.71	2.10	75	67	40	12	55	0.73	0.97	1.59
10×8...	102	53	16	14	39	0.41	0.40	0.81	75	35	5	2	33	0.15	0.20	0.56
11......	102	56	21	5	51	0.41	0.40	0.80	75	45	4	4	41	0.10	0.13	0.71
12......	102	81	60	23	58	1.03	1.01	1.61	75	64	46	10	54	0.85	1.13	1.60
9.......	102	70	46	13	57	0.81	0.79	1.26	75	53	16	4	49	0.33	0.44	0.97
10......	106	58	19	14	44	0.43	0.41	0.86	74	33	7	3	30	0.23	0.31	0.58
11×9...	106	58	24	10	48	0.50	0.47	0.87	74	41	14	5	36	0.39	0.53	0.81
12......	106	89	58	30	59	0.97	0.91	1.67	74	62	32	10	52	0.62	0.84	1.41
10......	106	49	17	5	44	0.39	0.37	0.67	74	38	7	5	33	0.21	0.28	0.68
11×10..	106	85	57	24	61	0.93	0.88	1.57	74	63	25	10	53	0.47	0.63	1.33
12......	100	75	58	25	50	1.16	1.16	1.68	74	61	40	11	50	0.80	1.08	1.51
11×11..	100	89	74	20	69	1.07	1.07	1.83	74	60	38	8	52	0.73	0.99	1.43
12......	100	93	95	24	69	1.38	1.38	2.12	74	65	59	9	56	1.05	1.42	1.80
12×12..	100	79	45	13	56	0.80	0.80	1.37	74	59	17	8	51	0.33	0.45	1.14

NOTE.—For explanation of columns, see note at end of Table IV., but here g and g^1 = 100f or $100f^1 \div a$ or a^1, respectively.

TABLE VI.—PART II.

TOTALS FOR BOYS AND GIRLS.

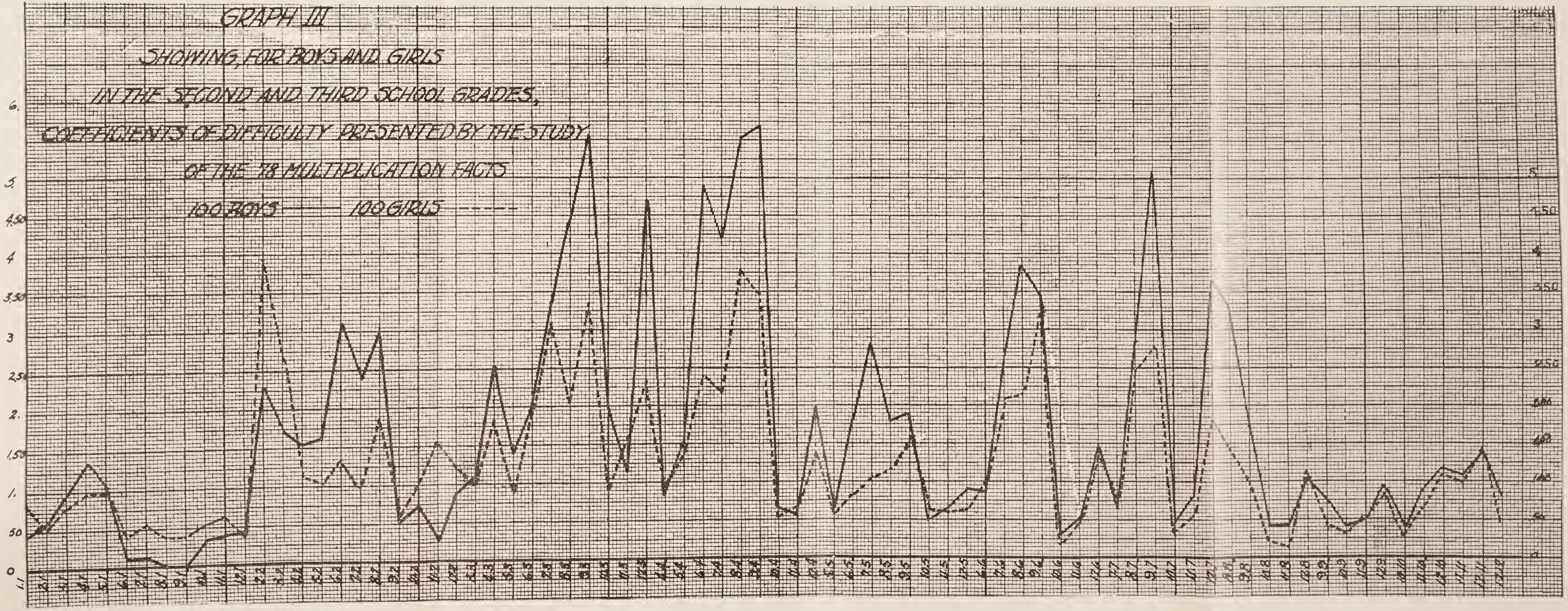
	A	B	C	D	E	F	G	H
3*	141	39	16	16	23	0.70	0.98	0.50
6	183	75	85	28	47	1.81	2.56	1.03
8×3	183	84	117	39	45	2.60	2.84	1.31
9	167	65	102	34	31	3.29	3.94	1.20
11	167	50	42	12	38	1.10	1.32	0.62
12	167	63	87	29	34	2.56	3.07	1.07
4	167	81	42	22	59	0.71	0.85	0.85
7	159	99	137	33	56	2.45	3.08	1 63
8×4	159	85	184	37	48	3.83	4.81	1.92
9	159	103	197	49	54	3.65	4.58	2.20
10	175	59	26	9	50	0.52	0.59	0.54
12	175	101	91	37	64	1.42	1.62	1.31
5	175	71	28	14	57	0.49	0.56	0.65
7×5	173	94	82	35	59	1.39	1.61	1.22
10	173	67	28	4	63	0.44	0.51	0.57
12	172	106	50	32	74	0.68	0.78	1.09
6	172	95	49	33	62	0.79	0.92	0.95
7	172	123	180	35	88	2.05	2.38	1.96
8×6	172	126	173	53	73	2.37	2.76	2.02
9	172	134	205	63	71	2.89	3.36	2.34
10	172	81	14	11	70	0.20	0.23	0.56
11	172	96	36	10	86	0.42	0.49	0.825
12	172	130	113	35	95	1.19	1.38	1.62
7	172	121	58	23	98	0.59	0.79	1.17
8	172	142	204	53	89	2.29	2.66	2.32
9×7	157	145	260	64	81	3.21	4.09	2.99
10	157	85	24	6	79	0.30	0.38	0.73
11	157	88	43	12	76	0.57	0.73	0.905
12	157	119	157	43	76	2.07	2.64	2.03
8	157	111	147	35	76	1.93	2.46	1.87
9	177	154	134	45	109	1.23	1.39	1.88
10×8	177	88	21	16	72	0.29	0.33	0.71
11	177	101	25	9	91	0.27	0.31	0.76
12	177	145	106	33	112	0.95	1.07	1.60
9	177	123	62	17	160	0.59	0.69	1.13
10	180	91	26	17	74	0.35	0.39	0.74
11×9	180	99	38	15	84	0.45	0.50	0.84
12	180	151	90	40	111	0.81	0.90	1.56
10	180	87	24	10	77	0.31	0.34	0.67
11×10	180	148	82	34	14	0.57	0.63	1.47
12	174	136	98	36	100	0.98	1.11	1.55

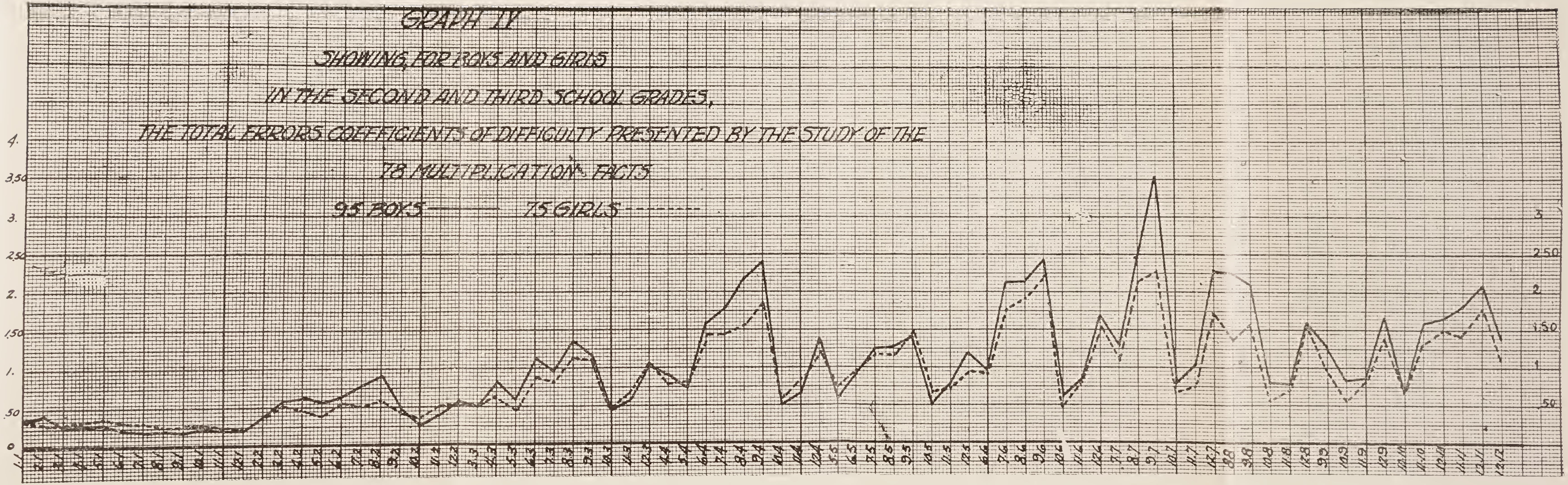
TABLE VI.—PART II.—*Continued.*

TOTALS FOR BOYS AND GIRLS.

	A	B	C	D	E	F	G	H
11×11	174	149	112	28	121	0.93	1.07	1.66
12	174	158	154	33	125	1.24	1.42	1.98
12×12	174	138	62	21	107	0.58	0.67	1.27

NOTE.—A, number of pupils; B, number of errors in preliminary tests; C, number of errors made in process of learning; D, number of errors in final tests; E (B — D), number of errors overcome; F (C ÷ E), co-efficients of difficulty for group tested; G (200F ÷ A), average co-efficients of difficulty. See g and g¹ of this table; H $\left\{\frac{B+C+D}{A}\right\}$, total errors co-efficients of difficulty.


GRAPH III
SHOWING, FOR BOYS AND GIRLS
IN THE SECOND AND THIRD SCHOOL GRADES,
COEFFICIENTS OF DIFFICULTY PRESENTED BY THE STUDY
OF THE 78 MULTIPLICATION FACTS
100 BOYS ——— 100 GIRLS - - - - -


GRAPH IV
SHOWING, FOR BOYS AND GIRLS
IN THE SECOND AND THIRD SCHOOL GRADES,
THE TOTAL ERRORS COEFFICIENTS OF DIFFICULTY PRESENTED BY THE STUDY OF THE
78 MULTIPLICATION FACTS
95 BOYS ——— 75 GIRLS - - - - -
4.
3.50
3.
2.50
2.
1.50
1.
.50
0

5. DISCUSSION OF THE RESULTS IN MULTIPLICATION.

The number, age, sex, and class distribution of the pupils as shown in Table V represent the conditions in regard to these factors which obtained throughout the entire study. The experiment ran through two terms of five months each, beginning in February and ending the following February. Hence, the conditions of the whole school year, such as vacations, holidays, promotions, &c., are all involved and argue for the validity of the results under circumstances that are practically duplicated from year to year in any small system of schools.

The variation in the number of pupils recorded in each group of combinations studied is due to the same process of elimination for absence as was described in the discussion of the addition results (XII 5, a).

All results in multiplication are reduced to the same base to facilitate comparison, and the same method of evaluation, or determining co-efficients of difficulty used in addition is again used for the multiplication (XII 5, a).

It must be borne in mind, as in the case of the addition graphs, that the results as shown by Graphs III and IV are of value only when the combinations are studied in the order of the group presentations—that is, progressively from group one to group sixteen. The order of the combinations in each group was always promiscuous. By this method of study it will be noted how, for instance, the effect of learning group one (1×1 to 5×1) is shown in learning the other combinations with 1, and how the general trend of difficulty at first upward soon begins to go downward and so continues as the size of the numbers increase. The "curves" show very clearly the peaks of difficulty and the depressions indicative of the lack of difficulty, in fact, a study of the graphs tells the whole story.

A further analysis of Graph III shows that in 15 out of the 78 cases the lines of the "curve" tend in opposite directions for boys and girls. Therefore, in 63 cases the trend of difficulty for boys and for girls is in the same direction, if not to the same values. A casual glance at this graph will show that the girls had less diffi-

culty in learning the multiplication combinations than the boys. If we calculate the averages of their respective co-efficients of difficulty, we fine that the average for boys is 1.61, while that for girls is only 1.18, or nearly 27% less. In other words, the girls learned the multiplication facts more readily than the boys by over a fourth.

Graph IV, showing the total errors co-efficients for boys and girls, presents a very interesting similarity of tendency for the two sexes. In this all the errors made in the course of the experiment are included. Here, again, we note an advantage in favor of the girls. A calculation of the average of the co-efficients of difficulty from columns in Table VI marked "h" and "h^1" gives for the boys 1.035, and for the girls .91. This shows that the girls made about 13% fewer errors throughout the drills than the boys.

These multiplication graphs should be of much assistance to the primary teacher who is confronted with the problem of teaching the multiplication facts. Aside from what they show of relative difficulty on their face, they should warn the teacher against an attempt to get along too rapidly in the early stages of the process. She should remember that here is the period for the development of power which is to serve her pupils well in their struggles with the larger numbers to come.

XIV. RELATIVE DIFFICULTY AS INDICATED BY ERRORS ALONE.

1. PLAN.

It was not deemed sufficient to study this problem from the two points of view only that have thus far been set forth. These have their value to the teacher in the *process of teaching* the combinations. It was also thought advisable to consider the problem that is presented to the teacher *after the ground has been gone over,* to point out to her the places where the tares of forgetfulness are most likely to have sprung up and to have choked out the newly planted memories. Through the kindness and co-operation of the Superintendent, Principals, and Third and Fourth Grade Teachers of

one of the large city systems of schools in New Jersey, the writer was able to get results from the examination of a much larger number of pupils than the foregoing experiments involved.

The plan was to give each third grade pupil in the city a written test, first, in the forty-five addition facts, and immediately thereafter, a test in the seventy-eight multiplication combinations. These tests were given during the last week of school preceding the summer vacation. It should be added here that the course of study in arithmetic provided for the teaching of all the tables to 12 × 12 by the end of the third year. The plan also proposed exactly the same tests to be given to the pupils of the fourth grade during the first week of school in September. This grade was, of course, selected with the idea of catching a maximum number of those who were tested in June and comparing the results of the two tests. This comparison would show the combinations most likely to be forgotten and thus provide the fourth grade teacher with a most valuable guide in directing the work of her number reviews.

2. NATURE OF THE TESTS.

With these ends in view tests in the forty-five facts of addition, promiscuously arranged, were given to 1,056 pupils, as indicated in Table VII, which shows the number of errors in the various combinations and the percentage of such errors. Table VIII shows the arrangement of these facts in the order of their difficulty, based upon the number of errors made in each.

The seventy-eight multiplication combinations, arranged promiscuously in exactly the same way as for the preliminary and final tests given in the first part of this investigation, were given to 1,215 pupils, as indicated in Tables IX and X.

The combinations, both in addition and multiplication, were typewritten and mimeographed. The directions to teachers which follow will explain more fully the nature of the tests, as well as explain the methods pursued in giving them.

3. DIRECTIONS FOR TEACHERS.

1. See that each pupil has a usable lead pencil.

2. To explain what is desired to your class: Take the addition blanks and hold them up before your pupils and say, "I have here some little examples in addition, and I want to see how *quickly* this class can write the answers after the = signs." (Show the method on one of the papers and on the board, as 12 + 12 = 24.) "You will each be given a sheet face down upon your desk. You are to fill in the blanks on the back of this first. (Show the place.) At the signal, 'turn papers,' you are to turn your sheet and write the answers, beginning with the left-hand column, and going down each in order. (Show how.) While you are doing this and at the end of every half minute, I shall say 'mark.' At that signal you will draw a line under the problem last finished. (Show how on the sheet held before the class.) Is there anyone who does not understand?" If all understand,

3. Distribute the papers as explained.

4. Have each pupil fill in the blanks on each sheet (name, grade, date). When all have finished this, say, "turn papers and write answers." You will then say "mark" at the end of each half minute.

5. As soon as each pupil has written all of the answers, he is to turn his sheet face down upon his desk and wait until the others have finished. Time should be given for all pupils to finish, but they are not to be told so. The maximum time given to the room for the exercise will be placed on the outside of each envelope.

6. The same directions should precede the giving of the work in multiplication, but the teacher will illustrate with 12 × 15 = 180.

7. Pupils must be instructed not to "count up" in addition, and not to "run down the row" in multiplication. They are to write the answers as fast as possible and *from memory*.

8. No explanations or interruptions are to take place after an exercise has begun.

9. When the tests have been finished, place the papers in the envelopes in which they are given you and return the same to the Principal.

10. As is indicated above, the addition test is to be given first. This is to be followed at once by the multiplication test.

TABLE VII.

SHOWING THE NUMBER OF ERRORS MADE IN THE ADDITION COMBINATIONS BY 1,056 CHILDREN.

	Boys.				Girls.				Grand Totals.	Per Cent. of Errors.
Grade	3	4	1 & 2	Totals.	3	4	1 & 2	Totals.		
No. Pupils...	332	115	55	502	338	112	54	554	1056	.....
1	4	1	1	6	8	2	1	11	17	1.6
2	12	....	2	14	6	....	1	7	21	2.0
3	8	....	1	9	6	2	2	10	19	1.8
4	3	1	2	6	7	1	1	9	15	1.4
5+1	5	....	2	7	7	1	......	8	15	1.4
6	6	....	2	8	6	3	......	9	17	1.6
7	5	....	3	8	8	3	1	12	20	1.9
8	4	....	8	12	7	....	......	7	19	1.8
9	3	....	4	7	2	3	1	6	13	1.2
2	3	1	1	5	4	....	......	4	9	0.8
3	5	1	3	9	6	1	1	8	17	1.6
4	7	....	3	10	3	....	3	6	16	1.5
5+2	5	1	......	6	4	....	3	7	13	1.2
6	3	....	4	7	8	1	3	12	19	1.8
7	6	1	2	9	9	3	3	15	24	2.3
8	5	2	1	8	3	....	1	9	12	1.1
9	2	....	1	3	6	3	3	12	15	1.4
3	2	2	1	5	2	1	......	3	8	0.7
4	5	2	2	9	8	....	1	9	18	1.7
5	5	4	3	12	7	1	6	14	26	2.5
6+3	3	2	2	7	5	2	6	13	20	1.9
7	8	3	6	17	10	4	6	20	37	3.5
8	10	4	8	22	7	7	5	19	41	3.9
9	14	3	4	21	14	5	3	22	43	4.1
4	3	3	......	6	....	....	2	2	8	0.7
5	5	1	3	9	8	1	2	11	20	1.9
6+4	8	3	4	15	10	5	4	19	34	3.2
7	14	3	5	22	19	4	3	26	48	4.5
8	13	6	6	25	4	5	3	12	37	3.5
9	14	5	5	24	17	5	5	27	51	4.8
5	4	1	......	5	1	....	3	4	9	0.8
6	10	5	3	18	7	1	7	14	32	3.0
7+5	10	11	9	30	11	6	8	26	56	5.3
8	19	11	5	35	23	4	6	33	68	6.4
9	15	4	7	26	13	7	3	23	49	4.6
6	10	1	1	12	5	1	2	8	20	1.9
7+6	13	6	5	24	16	4	6	26	50	4.7
8	17	8	11	36	14	9	7	30	66	6.2
9	16	8	7	31	22	14	5	51	82	7.8

TABLE VII.—*Continued.*

SHOWING THE NUMBER OF ERRORS MADE IN THE ADDITION COMBINATIONS BY 1,056 CHILDREN.

	Boys.				*Girls.*				Grand Totals.	Per Cent. of Errors
Grade	3	4	1 & 2	Totals.	3	4	1 & 2	Totals.		
No. Pupils...	332	115	55	502	338	112	54	554	1056	.....
7	8	2	4	14	5	1	......	6	20	1.9
8+7	18	8	8	34	19	7	9	35	69	6.5
9	29	8	8	45	27	8	10	45	90	8.5
8+8	13	3	4	20	12	2	3	17	37	3.5
9	30	12	14	56	23	5	11	39	95	9.0
9+9	5	4	6	15	7	2	3	14	29	2.7

The average number of errors made by each boy was 1.45; that for each girl 1.29. Here, again, we have a difference in favor of the girls, this time of 11%. A further comparison of the number of errors made by boys and girls, while not specially significant, shows a few points of interest. 8 + 4, with 25 errors, seemed to be much more difficult for the boys than for the girls, who made only 12 errors, while 9 + 6, with only 31 errors, seemed to be much easier for the boys than for the girls, who made 51 errors. 9 + 8, on the other hand, with 56 errors, seemed to be more difficult for the boys than for the girls, who made only 39 errors in this combination. In all the cases, however, the percentages of error differ but slightly.

TABLE VIII.

SHOWING ORDER OF DIFFICULTY OF ADDITION FACTS AND NUMBER OF ERRORS MADE BY 1,056 CHILDREN IN EACH FACT.

Combination.	No. Errors.	Combination.	No. Errors.
9 + 8 (6)	95	7 + 7	20
9 + 7 (1)	90	6 + 6	20
9 + 6 (3)	82	5 + 4	20
8 + 7 (8)	69	6 + 3	20
8 + 5 (2)	68	7 + 1	20
8 + 6 (omitted)	66	6 + 2	19
7 + 5 (10)	56	8 + 1	19
9 + 4 (omitted)	51	3 + 1	19
7 + 6 (15)	50	4 + 3	18
9 + 5 (5)	49	3 + 2	17
7 + 4 (13)	48	6 + 1	17
9 + 3 (4)	43	1 + 1	17
8 + 3 (14)	41	4 + 2	16
8 + 8 (12)	37	9 + 2	15
8 + 4 (7)	37	5 + 1	15
7 + 3 (11)	37	4 + 1	15
6 + 4 (omitted)	34	5 + 2	13
6 + 5 (9)	32	9 + 1	13
9 + 9	29	8 + 2	12
5 + 3	26	5 + 5	9
7 + 2	24	2 + 2	9
2 + 1	21	4 + 4	8
		3 + 3	8

NOTE.—The numbers in parenthesis indicate the order of the fifteen most difficult combinations according to the study made by Phelps (22) with eighth grade children.

1. *Discussion.*—In the above table it is interesting to note that the first 16 combinations in which errors are most frequent have each in them either a 9 or an 8 or a 7, the 8's appearing in the greatest number of cases, the 9's next, but in only 3 fewer cases, and the 7's next (cf. III. 1). It is also significant that the last four combinations on the list are doubles. That 1 + 1 stands so high in the number of errors appearing therein is rather startling, especially in the light of the fact that "1 + 1 = 1" was regarded as an error in "process" and not counted in this table. It is also interesting to observe the combinations which the same number of errors brings together; for example, 8+8, 8+4, 7+3; 7+7, 6 + 6, 5 + 4, 6 + 3, 7 + 1, &c.

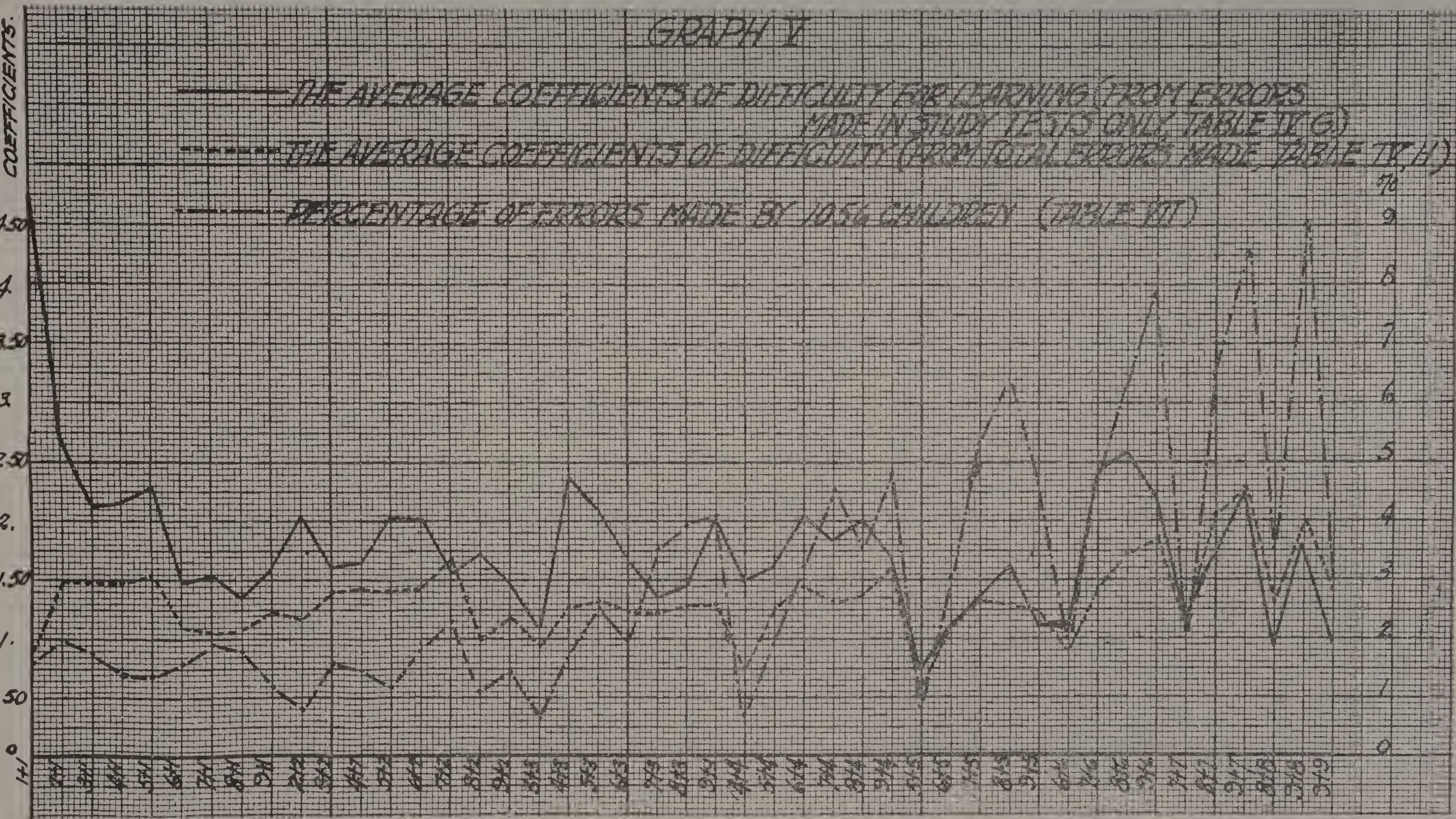
In order to show the relation of the number of errors in "process" to the number of other errors, records were kept for 720 third grade children as shown by the following table:

TABLE IX.

SHOWING THE NUMBER OF ERRORS MADE IN "PROCESS" (COLUMN "P") AS COMPARED WITH THE NUMBER OF OTHER ERRORS (COLUMN "O") MADE BY 720 THIRD GRADE CHILDREN.

Combination.	P	O	Combination.	P	O
1	28	12	4	57	3
2	39	18	5	57	13
3	41	14	6 + 4	54	18
4	42	38	7	50	33
5 + 1	38	12	8	40	17
6	40	12	9	39	31
7	40	13			
8	38	11	5	41	5
9	40	5	6	56	16
			7 + 5	46	21
2	28	7	8	53	52
3	47	11	9	33	28
4	51	10			
5	72	9	6	45	15
6 + 2	68	11	7 + 6	36	29
7	56	13	8	45	31
8	64	8	9	32	38
9	52	8			
			7	35	13
3	59	4	8 + 7	35	37
4	70	13	9	33	56
5	54	12			
6	63	8	8 + 8	26	23
7 + 3	52	18	9	27	53
8	48	17			
9	52	28	9 + 9	31	14

Discussion.—The large number of errors in process is doubtless due to the fact that the emphasis of the year's teaching had been upon the multiplication tables and not upon the addition facts. It might be argued that the total number of errors made in a given combination, counting both errors in process and other errors, gives a truer index of relative difficulty than the "other errors" as above used. In answer to this it will be observed that errors in process are generally most frequent when the multiplication of the numbers involved is easiest, and that these errors decrease as the difficulty in multiplication increases. The total of both errors in process and "other errors" would make 4 + 2 and 4 + 3 the most difficult of all the facts, which is manifestly not the case.


GRAPH V
THE AVERAGE COEFFICIENTS OF DIFFICULTY FOR LEARNING (FROM ERRORS MADE IN STUDY TESTS ONLY, TABLE IV G)
THE AVERAGE COEFFICIENTS OF DIFFICULTY (FROM TOTAL ERRORS MADE, TABLE IV, H)
PERCENTAGE OF ERRORS MADE BY 1054 CHILDREN (TABLE XII)
COEFFICIENTS.
4.50
4.
3.50
3.
2.50
2.
1.50
1.
50
0
%
9
8
7
6
5
4
3
2
1
0
1+1

It may be wondered how an error in process was counted for 2 + 2, in which the sum and the product are the same. This combination appeared in the group 7 + 9, 2 + 2, 3 + 1. If the first and last answers of the group were obtained by multiplication, it was presumed that the middle one was also multiplied instead of added.

Discussion.—The summary of results thus far obtained in addition is shown in the graphic representation (Graph V). Of these "curves," two should be especially helpful to the primary teacher: (1) the one giving the average co-efficients of difficulty for learning the combinations for both boys and girls; and (2) that showing the percentage of errors made by children. The former should indicate to the teacher the easy and the hard places as she proceeds in teaching the forty-five facts, while the latter shows the main points of attack in conducting review exercises with children who have presumably once learned the facts.

TABLE X.

SHOWING THE NUMBER OF ERRORS MADE IN THE MULTIPLICATION COMBINATIONS BY 1,215 CHILDREN.

	Boys.					*Girls.*					Grand Totals.	Per Cent. of Errors.
Grade	4	3	2 & 3	5 & 3	Totals.	4	3	2 & 3	5 & 3	Totals.		
No. Pupils.	133	334	61	58	586	135	394	42	58	629	1215	....
*4........	6	13	6	18	43	6	19	4	4	33	76	6.2
6........	4	15	6	29	54	12	23	3	10	48	102	8.4
8×3.....	10	31	7	17	65	19	46	4	17	86	151	12.4
9........	13	26	10	23	72	22	47	8	20	97	169	13.9
12........	21	46	6	16	89	19	61	2	12	94	183	15.0
4........	9	16	4	7	36	10	21	2	9	42	78	7.2
8........	22	54	6	37	119	23	59	15	19	116	235	19.3
9×4.....	39	63	7	38	147	31	76	16	22	145	292	24.0
10........	2	1	......	5	8	4	19	......	......	23	31	2.5
12........	25	60	11	26	122	20	82	8	18	128	250	20.6
5........	...	7	1	4	12	4	14	......	4	22	34	2.8
6........	14	27	6	16	63	16	39	2	18	75	138	11.9
7........	14	47	10	20	91	17	50	4	19	90	181	14.9
8........	16	28	12	19	75	13	31	3	15	62	137	11.3
9×5.....	36	16	7	28	87	25	32	5	19	81	168	13.8
10........	9	12	......	5	26	3	28	......	1	32	58	4.8
11........	8	22	1	13	54	13	34	......	12	59	113	9.3
12........	39	59	5	28	131	31	90	2	17	140	271	22.3
6........	23	26	4	15	68	20	26	4	11	61	129	10.6
7........	34	64	11	28	137	30	85	8	25	148	285	23.4
8........	42	77	13	30	162	45	100	11	24	180	342	28.2
9×6.....	47	90	15	39	191	45	113	10	31	199	390	32.1
10........	11	20	1	7	39	12	22	1	5	40	79	6.5
11........	29	38	5	15	87	16	30	......	11	57	144	11.8
12........	35	90	11	35	171	36	114	8	32	190	361	28.9
7........	31	61	3	31	126	41	81	2	18	142	268	22.1
8........	54	95	11	39	199	65	134	8	29	236	435	35.8
9×7.....	53	103	12	46	214	65	131	15	30	241	455	37.4
10........	13	19	1	9	42	14	22	1	7	44	86	7.1
11........	22	35	1	11	69	17	39	1	11	68	137	11.3
12........	54	103	14	32	203	45	147	8	35	235	438	36.0
8........	51	93	5	26	175	49	109	9	19	186	361	29.7
9........	54	95	18	38	205	57	118	13	29	217	422	34.7
10×8.....	13	14	2	8	37	11	34	1	2	48	85	7.0
11........	27	49	3	14	93	19	46	......	9	74	167	13.7
12........	59	116	11	33	219	52	155	5	29	241	460	37.9

TABLE X.—*Continued.*

SHOWING THE NUMBER OF ERRORS MADE IN THE MULTIPLICATION COMBINATIONS BY 1,215 CHILDREN.

	Boys.					*Girls.*					Grand Totals.	Per Cent. of Errors.
Grade	4	3	2 & 3	5 & 3	Totals.	4	3	2 & 3	5 & 3	Totals.		
No. Pupils.	133	334	61	58	586	135	394	42	58	629	1215	....
9........	34	62	4	22	122	43	74	6	18	141	263	21.6
10×9.....	11	19	2	7	39	11	40	1	3	55	94	7.7
11........	22	45	3	12	82	25	61	2	11	99	181	14.9
12........	57	101	5	29	192	50	143	4	28	225	417	34.4
10........	25	61	2	6	94	32	107	......	8	147	241	19.8
11×10....	79	170	11	32	292	69	240	5	32	346	638	52.5
12........	61	136	5	28	230	67	209	6	30	312	542	44.6
11×11....	80	232	4	35	351	85	268	2	29	384	735	60.5
12........	78	181	6	40	305	83	234	6	27	350	655	54.0
12×12....	48	119	1	23	191	56	163	......	15	234	425	35.0

* See note at bottom of Table VI.—Part I.

It should be noted in connection with the above figures that the results from the fourth grade pupils were secured during the first week of school in September, so likewise those from the "5 & 3" grades. All other results were obtained in June.

TABLE XI.

SHOWING ORDER OF DIFFICULTY OF THE MULTIPLICATION COMBINATIONS AND THE NUMBER OF ERRORS MADE BY 1,215 CHILDREN IN EACH FACT.

Combination.	No. Errors.	Combination.	No. Errors.
11 × 11	735	6 × 3	102
12 × 11	655	11 × 3	99
11 × 10	638	10 × 9	94
12 × 10	542	10 × 7	86
12 × 8	460	10 × 8	85
9 × 7 (1)	455	12 × 2	81
12 × 7	438	10 × 6	79
8 × 7 (4)	435	4 × 4	78
12 × 12	425	4 × 3	76
9 × 8 (7)	422	7 × 3	71
12 × 9	417	10 × 5	58
9 × 6 (2)	390	8 × 2	58
8 × 8 (3)	361	5 × 4	55
12 × 6	361	6 × 2	50
8 × 6 (5)	342	5 × 3	46
9 × 4	292	11 × 2	46
7 × 6 (6)	285	1 × 1	41
12 × 5	271	9 × 2	39
7 × 7 (8)	268	10 × 3	38
9 × 9 (10)	263	7 × 2	38
12 × 4	250	5 × 5	34
10 × 10	241	4 × 2	32
8 × 4 (9)	235	10 × 4	31
7 × 4	192	10 × 2	31
12 × 3	183	11 × 1	31
11 × 9	181	4 × 1	31
7 × 5	181	3 × 1	28
9 × 3	169	5 × 2	26
9 × 5	168	3 × 3	25
11 × 8	167	9 × 1	22
8 × 3	151	3 × 2	21
11 × 6	144	7 × 1	21
6 × 5	138	6 × 1	21
11 × 7	137	12 × 1	20
8 × 5	137	5 × 1	20
6 × 4	133	2 × 1	20
11 × 4	131	2 × 2	18
6 × 6	129	8 × 1	18
11 × 5	113	10 × 1	12

NOTE.—The numbers in parenthesis indicate the order of the ten most difficult combinations according to the study made by Max Döring (6).

Discussion.—Table X shows the errors which were made in the same kind of a test in multiplication as was given in addition. It represents the tabulation of over 13,000 errors. The average number of errors for each boy was 10.8, that for each girl

11.2. As this was essentially a test in power of retention, the boys seemed to possess that power in the case of multiplication 2.7% in excess of the girls. A comparison of the details shows little or no difference between the boys and the girls until 11 × 8 is reached. From there to 12 × 12 the percentage of errors made by the girls is constantly higher than that made by the boys. A comparison of the tables in which 7, 8, 9, and 12 appear as the first factor shows practically no difference until 12 is reached. Here the difference is marked, being nearly 10% in the favor of the boys. With the 10th and 11th rows the advantage is also with the boys. With the 6th and below the advantage is slightly with the girls, but it is so small as to be almost disregarded. It would seem, therefore, that boys have a retentive power for larger products in excess of that possessed by the girls.

As has been already explained, Professor Döring's experiment extended only to 10 × 10. A comparison in Table XI with the order established by his investigations shows that only 9 × 4 and 10 × 10 of the numbers coming under his consideration crept into his ten most difficult combinations, and that only the first place of difficulty in the two experiments agree, though a general agreement as to relative order of the other places, with the exception of his 4th and 7th places of difficulty, is to be noted. One would think that 10 × 10 would have very few errors, but the combination appeared so often as "10 × 10 = 110." The "10" sequence established in the part of the problem carried over with disastrous results into the product.

2. ERRORS IN PROCESS.

Errors in process were not counted in the multiplication tests just described, for the same reasons given for not counting them in the addition tests. However, in order to show the relation which the number of such errors bore to other errors the following table constituting records for 728 third grade pupils is added:

TABLE XII.

SHOWING THE NUMBER OF ERRORS MADE IN "PROCESS" (COLUMN "P") AS COMPARED WITH THE NUMBER OF OTHER ERRORS (COLUMN "O") MADE BY 728 THIRD GRADE CHILDREN.

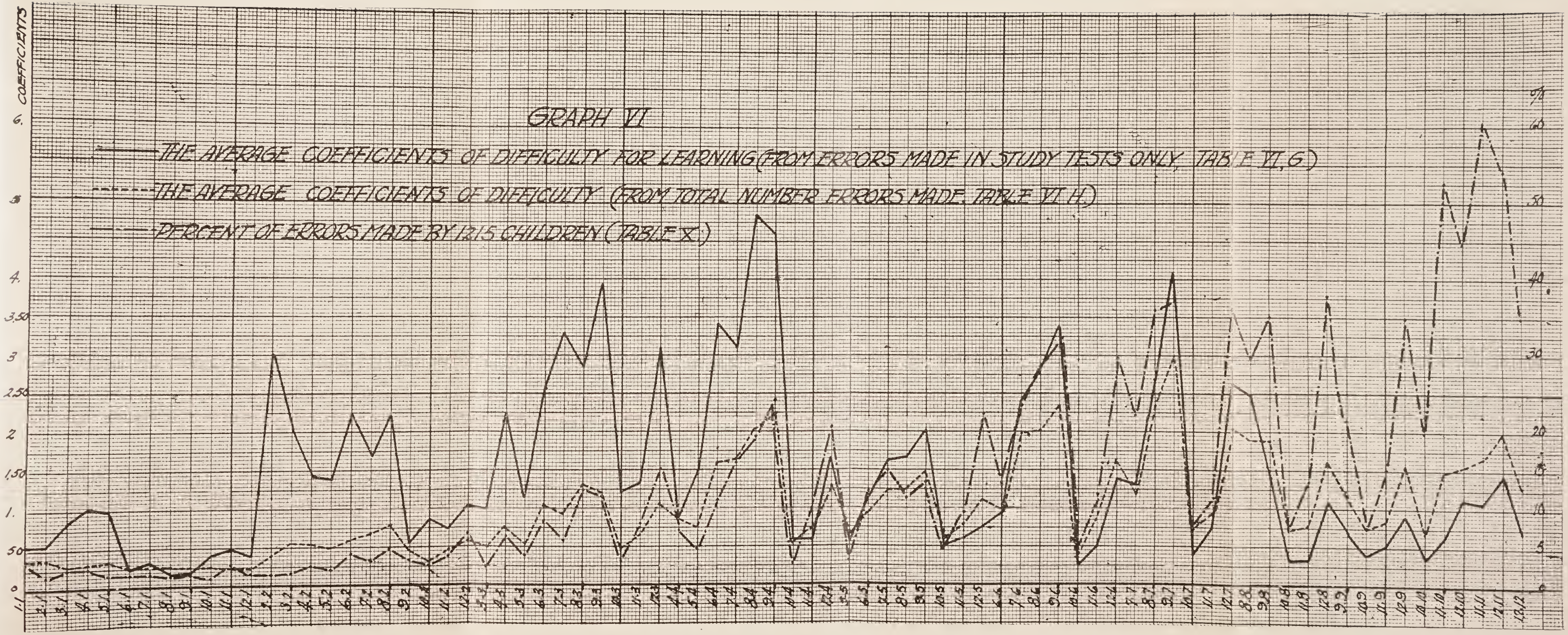
Combination.	P	O
1	94	31
2	43	16
3	33	23
4	22	26
5	29	12
6 × 1	27	16
7	26	15
8	25	8
9	22	15
10	22	8
11	20	21
12	24	15
2	25	8
3	34	6
4	35	14
5	24	13
6	29	14
7 × 2	21	13
8	23	19
9	23	7
10	21	17
11	21	25
12	24	44
3	68	12
4	31	32
5	27	20
6	29	38
7	26	28
8 × 3	24	77
9	24	73
10	20	18
11	17	53
12	22	107
4	61	37
5	23	28
6	19	62
7	20	95
8 × 4	24	113
9	18	139
10	20	20
11	18	82
12 × 4	23	142
5	40	21
6	25	66
7	22	97
8	20	59
9	20	48
10	19	40
11	19	56
12 × 5	20	149
6	38	52
7	22	149
8	23	177
9	24	203
10	21	42
11	22	68
12 × 6	24	204
7	42	142
8	24	229
9 × 7	22	234
10	21	41
11	20	74
12	20	250
8 × 8	37	202
9	22	213
10	22	48
11	20	95
12	21	271
9	27	136
10	19	59
11	17	106
12 × 9	20	244
10 × 10	40	168
11 × 10	21	410
12 × 10	22	345
11 × 11	53	500
12 × 11	19	415
12 × 12	42	282

Discussion.—An examination of the above table shows that the predominance of errors is found in the doubles, especially in 1×1. The small number in 2×2, compared with the other doubles, is due, of course, to the fact that the product and the sum of this combination are the same, so it was frequently impossible to tell from the context whether the pupil had added or multiplied.

Here again it was thought best to count only errors which could not be subsumed under the head of process or of inversions. The number of inversions was so small, however, as to be neglected, there being only 22 in all, 8 of these being in 9×5 and 4 in 9×6, which, of course, is significant.

Discussion.—The summary of the results so far obtained in multiplication is shown in the above graphic representation (Graph VI). The most interesting thing noticed in a study of the three "curves" recorded in this figure is their similarity and their disparity. Up to 10×5 the similarity between the values of the total errors co-efficients of difficulty and the per cent. of errors made by 1,215 children is very striking indeed. At this point a diversion begins which constantly increases to the end of the scale, yet coinciding in direction in almost every point. Up to 10×3 the per cent. values are almost invariably lower than the total errors co-efficients of difficulty values. From 10×3 to 10×5 there seems to be a struggle for supremacy, as it were, in which first one of them then the other succeeds, but after 10×5 the per cent. values are almost invariably higher. This, of course, results from the relation which exists between the two kinds of co-efficients shown in the figure, the learning co-efficient of difficulty having in it the same elements, though differently treated, as are contained in the total errors co-efficient.

The most significant relationship, however, lies in the comparison of the learning co-efficients and the per cent. of error. Up to 10×4 the former are invariably higher than the latter; then comes again a struggle for supremacy, as it were, lasting to about 10×6. After that point, with one notable exception, that of 9×7, the learning co-efficients are lower to the end of the series. While their divergence constantly increases, the value


GRAPH VI
THE AVERAGE COEFFICIENTS OF DIFFICULTY FOR LEARNING (FROM ERRORS MADE IN STUDY TESTS ONLY, TABLE VI, G.)
THE AVERAGE COEFFICIENTS OF DIFFICULTY (FROM TOTAL NUMBER ERRORS MADE, TABLE VI H.)
PERCENT OF ERRORS MADE BY 1215 CHILDREN (TABLE X.)
COEFFICIENTS
%

tendencies in both cases remain, for the most part, in the same direction.

The high value of the learning co-efficient in the early part of the scale, even for what we usually regard as the easy combinations, is commensurate with the time taken for mastery and the number of errors made in the process of mastery. The low values at the end of the scale are likewise commensurate with the same factors, and demonstrate beyond question the development of power in learning the combinations in the early part of the number scale.

The relation between the two "curves" would seem to show that facts easily mastered are easily forgotten. Compare the "curves" at the points 8×4, 9×4, 9×6, 9×7, 12×8, 12×9, 11×10, 11×11, and 12×11.

As in the case of Graph V for addition the learning co-efficients of difficulty here taken in the order and in the groups in which the combinations were presented should be especially helpful to the teacher in teaching the facts, and likewise the per cent. of errors "curve" should be of great value to the teacher in directing the review work of her class.

XV. THE EFFECT OF SUMMER VACATION UPON MEMORY FOR THE COMBINATIONS.

In order to determine the effect of ten weeks of summer vacation upon the memory for number combinations, the same tests were given during the first week of school in September as were given during the last week of school in June, but to classes one in advance of those in which the test in June was given. This gave results only for such pupils as were promoted. Five hundred and fifty-eight pupils were involved in the tests, distributed as follows: 516 third grade, 20 second grade, and 22 first grade. The entire results are given below:

TABLE XIII.

SHOWING THE NUMBER OF ERRORS IN ADDITION MADE BY 558 PUPILS BEFORE AND AFTER SUMMER VACATION.

	Before Vacation.				After Vacation.				
Grade	3	2	1	Total.	3	2	1	Total.	Increase.
1	2	1	.....	3	2	4	.....	6	3
2	5	2	1	8	3	6	.....	9	1
3	4	3	.....	7	9	6	.....	15	8
4	4	1	.....	5	2	5	.....	7	2
5+1	8	1	.....	9	1	5	.....	6	—3
6	5	.....	.....	5	7	5	.....	12	7
7	5	1	.....	6	5	3	1	9	3
8	7	1	.....	8	2	5	.....	7	—1
9	3	2	.....	5	5	3	1	9	4
2	2	.....	.....	2	2	2	.....	4	2
3	6	1	.....	7	7	4	1	12	5
4	5	2	.....	7	2	2	2	6	—1
5	6	.....	1	7	6	6	2	14	7
6+2	5	1	1	7	6	6	2	14	7
7	6	2	.....	8	7	4	1	12	4
8	5	1	.....	6	7	4	1	12	6
9	5	2	.....	7	6	7	4	17	10
3	1	0	.....	1	3	4	.....	7	6
4	6	1	1	8	4	5	3	12	4
5	12	1	1	14	10	7	1	18	4
6+3	3	1	1	5	12	5	4	21	16
7	16	2	1	19	18	5	3	26	7
8	11	3	1	15	18	6	1	25	10
9	14	2	1	17	19	8	4	31	14
4	.....	.....	.....	0	4	1	.....	5	5
5	6	2	.....	8	9	6	5	20	12
6+4	10	1	.....	11	14	6	1	21	10
7	18	2	2	22	24	5	3	32	10
8	9	2	3	14	13	5	1	19	5
9	23	1	3	27	25	9	7	41	14
5	2	1	.....	3	4	1	.....	5	2
6	12	1	1	14	17	8	4	29	15
7+5	14	2	1	17	34	5	5	44	27
8	39	4	.....	43	30	5	4	39	—4
9	18	2	.....	20	29	10	2	41	21
6	5	.....	1	6	5	4	1	10	4
7+6	22	3	3	28	23	9	5	37	9
8	21	1	4	26	36	8	3	47	21
9	29	1	1	31	38	10	6	54	23

TABLE XIII.—*Continued.*

SHOWING THE NUMBER OF ERRORS IN ADDITION MADE BY 558 PUPILS BEFORE AND AFTER SUMMER VACATION.

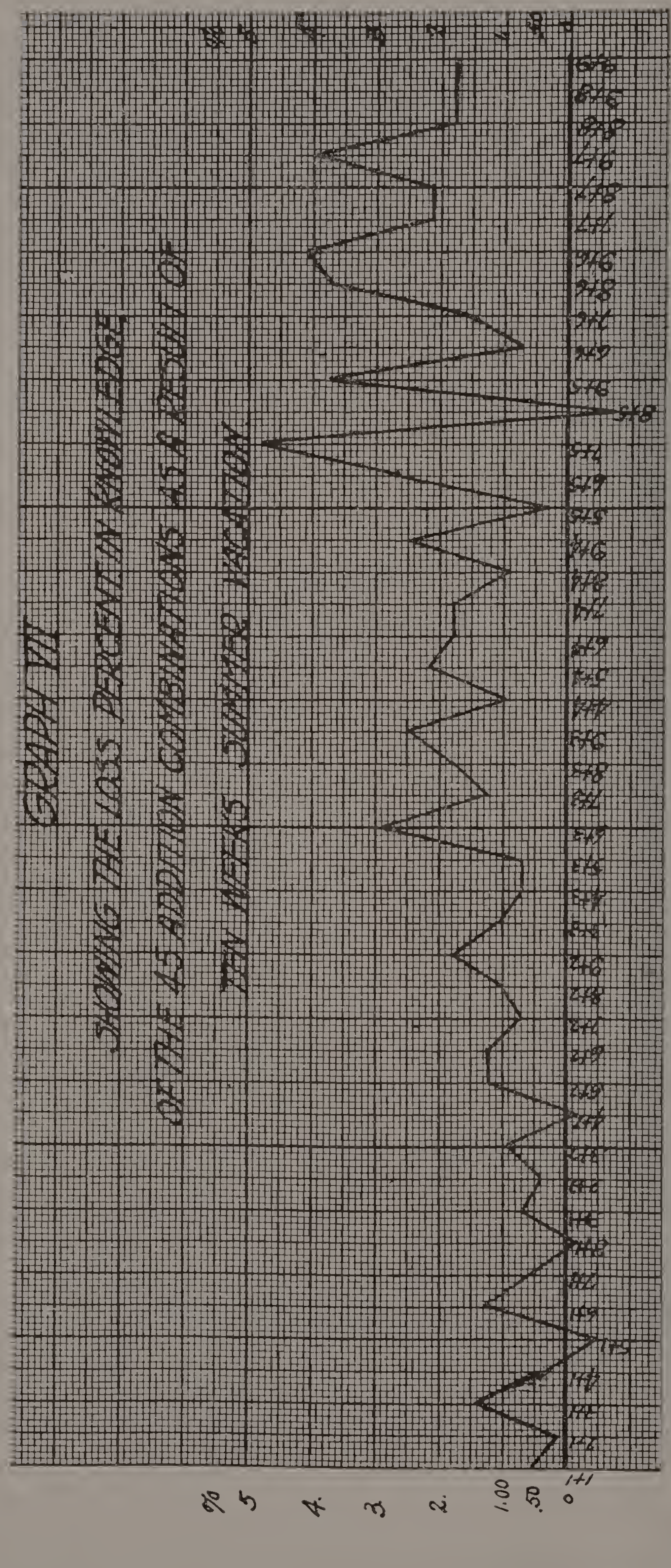
	Before Vacation.				*After Vacation.*				
Grade	3	2	1	Total.	3	2	1	Total.	Increase.
7	3	.....	1	4	7	7	2	16	12
8+7	29	2	3	34	32	9	5	46	12
9	30	4	2	36	45	9	4	58	22
8+8	10	1	1	12	12	7	3	22	10
9	38	4	3	45	36	13	6	55	10
9+9	9	1	.....	10	13	6	1	20	10

TABLE XIV.

SHOWING ADDITION FACTS ARRANGED IN THE ORDER OF THE PER CENT. FORGOTTEN DURING THE SUMMER VACATION.

Combination.	Loss Per Cent.	Combination.	Loss Per Cent.
7 + 5	4.85	5 + 2	1.3
9 + 6	4.1	6 + 1	1.3
9 + 7	3.9	3 + 3	1.1
8 + 6	3.8	8 + 2	1.1
9 + 5	3.8	8 + 4	0.9
6 + 3	2.9	4 + 4	0.9
6 + 5	2.7	3 + 2	0.9
9 + 4	2.5	6 + 6	0.7
9 + 3	2.5	5 + 3	0.7
8 + 7	2.15	4 + 3	0.7
7 + 7	2.15	7 + 2	0.7
5 + 4	2.15	7 + 1	0.7
9 + 9	1.8	7 + 1	0.5
9 + 8	1.8	1 + 1	0.5
8 + 8	1.8	5 + 5	0.4
7 + 4	1.8	2 + 2	0.4
6 + 4	1.8	4 + 1	0.4
8 + 3	1.8	2 + 1	0.2
9 + 2	1.8	4 + 2	—0.2
7 + 6	1.6	8 + 1	—0.2
3 + 1	1.4	5 + 1	—0.5
7 + 3	1.3	8 + 5	—0.7
6 + 2	1.3		

NOTE.—The negative values indicate gain instead of loss. Percentages are calculated on the basis of the number of errors possible.


GRAPH VII
SHOWING THE LOSS PERCENT IN KNOWLEDGE
OF THE 45 ADDITION COMBINATIONS AS A RESULT OF
TEN WEEKS SUMMER VACATION.
%
5
4.
3
2.
1.00
.50
0

Discussion.—In the study of Graph VII and Table XIV a word of warning is necessary. They alone should not be made the basis of any teaching practice. One point in the figure and table will serve to illustrate. It will be noted that at 8 + 5 there is an actual gain in knowledge, indicated by the "curve's" falling below the zero line, while the per cent. of loss for 7 + 5 is the maximum for the whole figure. This does not mean that in the process of review of the facts of addition that much emphasis should be placed upon the 7 + 5 fact and little or no attention should be given to the 8 + 5. The same point is to be noted with the high and the low values throughout the "curve." In the case above cited only 17 errors appeared for 7 + 5 in the June tests while there were 43 for 8 + 5. Greater opportunity for loss was, therefore, presented by the former than for the latter. In the September tests there actually occurred five fewer errors for 8 + 5 than for 7 + 5. This figure and table should be studied in connection with Table VII and the corresponding part of Graph V.

The value of Table IX and Graph VII lies in this fact: they point out the relative stability of the combinations from the standpoint of memory.

TABLE XV.

SHOWING THE NUMBER OF ERRORS MADE BY 530 THIRD GRADE CHILDREN IN THE MULTIPLICATION COMBINATIONS BEFORE AND AFTER THE SUMMER VACATION.

Combinations.	Before.	After.	Increase.	Combinations.	Before.	After.	Increase.	Combinations.	Before.	After.	Increase.
1.......	22	28	6	8×3....	55	85	30	11×6....	42	76	34
2.......	11	12	1	9.......	58	81	23	12.......	130	180	50
3.......	12	5	—7	10.......	17	26	9				
4.......	19	3	—16	11.......	33	52	19	7.......	91	141	50
5.......	7	6	—1	12.......	70	93	23	8.......	151	249	98
6×1....	12	8	—4					9×7....	166	256	90
7.......	9	5	—5	4.......	30	27	—3	10.......	33	35	2
8.......	3	5	2	5.......	19	30	11	11.......	46	69	23
9.......	10	9	—1	6.......	47	69	22	12.......	164	226	62
10.......	4	6	2	7.......	71	105	34				
11.......	16	17	1	8×4....	67	121	54	8.......	143	203	60
12.......	8	8	0	9.......	102	161	59	9×8....	151	253	102
				10.......	7	21	14	10.......	31	28	—3
2.......	6	9	3	11.......	51	41	—13	11.......	59	81	22
3.......	6	17	11	12.......	103	130	27	12.......	165	260	95
4.......	12	7	—5								
5.......	6	12	6	5.......	9	13	4	9×9....	91	184	93
6.......	7	25	18	6.......	45	69	24	10.......	42	42	0
7×2....	6	18	12	7.......	74	80	6	11.......	70	90	20
8.......	13	35	22	8×5....	39	76	37	12.......	146	270	124
9.......	4	20	16	9.......	44	104	60				
10.......	13	13	0	10.......	26	22	—4	10.......	105	117	12
11.......	15	17	2	11.......	29	39	10	11×10...	263	318	55
12.......	24	40	16	12.......	96	144	48	12.......	227	290	63
3.......	6	11	5	6.......	34	71	37	11×11...	326	356	30
4×3....	22	30	8	7.......	108	151	43	12.......	278	341	63
5.......	14	21	7	8.......	124	170	46				
6.......	30	49	19	9×6....	142	196	54	12×12...	188	241	53
7.......	18	46	28	10.......	23	40	17				

NOTE.—Average time taken for test before vacation, 6.3 minutes; after vacation, 5.9 minutes.

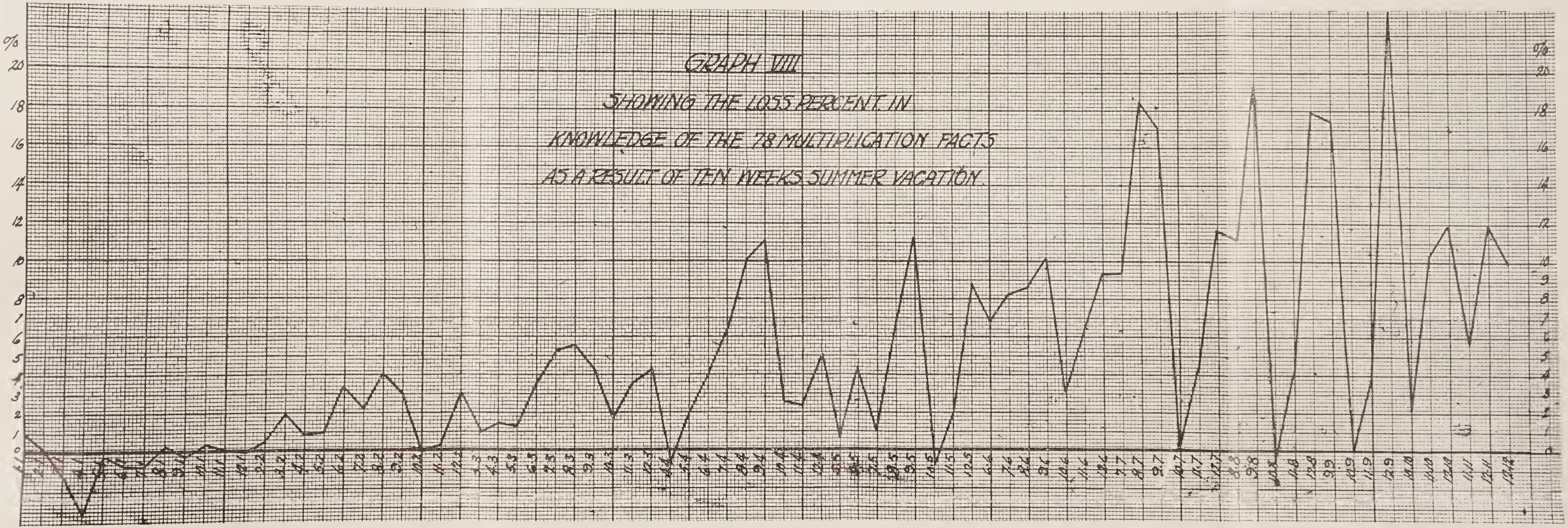

GRAPH VIII
SHOWING THE LOSS PERCENT IN
KNOWLEDGE OF THE 78 MULTIPLICATION FACTS
AS A RESULT OF TEN WEEKS SUMMER VACATION.

TABLE XVI.

SHOWING MULTIPLICATION COMBINATIONS IN THE ORDER OF THE PER CENT. FORGOTTEN DURING THE SUMMER VACATION.

Combination.	Loss.	Combination.	Loss.	Combination.	Loss.
	%		%		%
12 × 9	23.4	8 × 3	5.65	7 × 5	1.1
9 × 8	19.2	7 × 3	5.3	5 × 2	1.1
8 × 7	18.5	12 × 4	5.1	1 × 1	1.1
12 × 8	17.9	6 × 5	4.5	3 × 3	0.9
9 × 9	17.5	11 × 7	4.3	5 × 5	0.75
9 × 7	17.0	12 × 3	4.3	2 × 2	0.6
12 × 11.......	11.9	9 × 3	4.3	10 × 7	0.4
12 × 10.......	11.9	11 × 8	4.15	11 × 2	0.4
12 × 7	11.7	6 × 4	4.15	10 × 1	0.4
8 × 8	11.3	8 × 2	4.15	8 × 1	0.4
9 × 5	11.3	11 × 9	3.8	11 × 1	0.2
9 × 4	11.1	11 × 3	3.6	2 × 1	0.2
11 × 10.......	10.4	6 × 3	3.6	10 × 9	0
9 × 6	10.2	6 × 2	3.4	10 × 2	0
8 × 4	10.2	10 × 6	3.2	12 × 1	0
12 × 12.......	10.0	12 × 2	3.0	4 × 1	—0.2
7 × 7	9.45	9 × 2	3.0	5 × 1	—0.2
12 × 6	9.45	12 × 4	2.6	10 × 8	—0.6
12 × 5	8.85	10 × 10........	2.3	4 × 4	—0.6
8 × 6	8.7	7 × 2	2.3	10 × 5	—0.75
7 × 6	8.3	5 × 4	2.1	7 × 1	—0.75
6 × 6	7.0	3 × 2	2.1	6 × 1	—0.75
8 × 5	7.0	11 × 5	1.9	4 × 2	—0.9
11 × 6	6.4	10 × 3	1.7	3 × 1	—1.3
7 × 4	6.4	4 × 3	1.5	11 × 4	—2.45
11 × 11.......	5.65	5 × 3	1.3	4 × 1	—3.2

NOTE.—The negative values indicate gain instead of loss. Percentages are calculated on the basis of number of errors possible.

Discussion.—It is interesting to compare the order in Table XVI with that arranged on the basis of number of errors made by 1,215 pupils, shown in Table XI. It will be observed in this comparison that the 21 combinations which stand highest in Table XVI are to be found among the first 23 combinations that appear first in Table XI. The next group of 10 in Table XVI is to be found, with a few exceptions, in the corresponding part of Table XI, and so on, but there is no other similarity in the tables that is apparent.

The ratio between the per cent. of loss to the per cent. of error is far from constant. If the value representing the lack of stability of the number facts bore a direct relation to the number of errors, this relation should be constant throughout.

The same warning to which attention was called in regard to the application of Graph VII must also be issued here with reference to the use of Graph VIII. It must be studied in connection with Graph VII, especially with reference to that part showing the per cent. of errors made by 1,215 children.

XVI. VACATION SCHOOLS.

1. RESULTS.

Of the 558 children included in the addition tests some over 100 attended school from one to four weeks during the summer. That this fact did not materially, if at all, affect the results for the whole group is shown by the following: The ratio between the number of errors made in June to the number made in September by 112 pupils selected at random from those who did not attend vacation school was found to be .896; while a similar ratio for 112 children who attended vacation school was .815. What would have been the ratio for those who went to vacation school, if they had not attended is impossible to say, but whatever may have been the effect of instruction during the short summer period, it was not sufficient in this particular to bring the standing of those who took advantage of it up to that made by those who did not.

In the comparative multiplication tests the ratio of the number of errors in June to the number of errors in September for 112 pupils, selected at random from those not attending vacation school, was found to be .735; while a similar ratio for the same number of children who did attend vacation school was .716, the results again being in favor of those children who did not go to school in summer.

Since all the pupils in each case were promoted to the next higher class, we might presume that the vacation school was made up largely of those who may have stood low in their class. In any case, the fact that one-fifth of the pupils did attend school from one to four weeks during the summer cannot greatly influence the results of these comparisons.

XVII. RELATIVE DIFFICULTY AS INDICATED BY SKILL IN MANIPULATION.

1. PLAN AND NATURE OF THE TESTS.

There remains but one other important test for the determination of relative difficulty, and that is the test of skill in the use of the combinations. To determine this the forty-five addition combinations were divided into nine groups of five facts each. These groups contained the same combinations as the groups used for teaching the facts in the first part of this investigation. The seventy-eight multiplications were divided into sixteen groups in like manner, except that the fourth group was inadvertently made to contain four combinations instead of the second group.

For conducting this part of the experiment the combinations were arranged across a sheet of paper as shown below, the same being a reproduction of Group 3 for addition:

(3)	2	7	2	4	2	7	2	2	4	2	3	2	6
	5	2	3	2	6	2	3	5	2	6	2	4	2
	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—

There were usually about 15 examples in all, the purpose being to have more than any one pupil could do during the time allotted for each group. It will be observed that after the first five facts in the row the same facts are repeated in different order and in reverse form and direct form, so that no order or answers could assist the pupil in writing the products more rapidly than his normal rate would be for the first five sums or products, as the case might be. The nine groups were thus arranged for the first test in regular order down the page. For the second test the groups were arranged in reverse order, and for the third test the following order was given: 6, 7, 8, 9, 1, 2, 3, 4, 5. The object here was to equalize the effects of fatigue. Across the top of each sheet was typewritten, "Write the sums of the following numbers from left to right."

On the back of each sheet were mimeographed the answers of the various combinations in the order in which they appeared on the face, with a short line drawn under each.

These tests were given to the A and B classes of the third and fourth grades.

2. THE OBJECT OF THE TESTS.

The object of the experiment was first to determine the time required to write the sums of the various groups, or the products, as the case required, on the face of the sheets, then to determine the time required to copy the answers on the back of the sheets. By subtracting the time for the latter almost purely mechanical part from the former, a time representing more nearly the net time required to think the results would be obtained. That group requiring, therefore, the longest time to think out was adjudged the most difficult.

3. MANNER OF CONDUCTING THE TESTS.

The experiments were conducted personally by the writer and in the following manner: To each pupil there was given a sheet of white paper on which were mimeographed the combinations as above described. Covering each white sheet was a blank yellow sheet of the same size. This completely hid from view the work to be done. It was then explained that on the white sheet there was a number of rows of little examples in addition (or multiplication), the answers to which were to be written as fast as possible on the giving of a certain signal. The arrangement of the examples was shown on a sheet in the hands of the experimenter. He then showed how the yellow sheet was to be pulled down to disclose one row at a time at the signal, "papers down," and how on the signal of "write," given immediately after, the pupils were to write as many of the answers as they could until the experimenter said "stop," after which the papers were to be pushed up over the row just written. This was to prevent additions or corrections. At the next signal "papers down," the next row below was to be exposed and as many answers in that row were to be written as time permitted. A rest of from five to ten seconds was given between the writing of each row. When the last row had been written, the papers were turned over and all the answers on the back of the sheet were exposed at once. It was then explained that these were to be copied in the place indicated by the mark under each and from left to right across the page, the work to begin at a certain signal. The signals used for this part of the test were as follows: (1) "Get ready," which meant put the pencil where the first answer was to be written; (2) "write," which meant that as many of the answers as possible were to be copied until the third signal, "stop," was given. Five or ten seconds' rest was allowed between each row. The order

of these groups of answers was made to follow the order of the groups of examples on the other side of the sheet.

For the third grade addition tests 15 seconds was the time allowed for writing the sums of each group, and 5 seconds for the answers. For the fourth grade 10 seconds and 5 seconds, respectively, were allowed.

4. METHOD OF EVALUATING RESULTS.

The number of sums or products written, and the number of answers copied, were arranged in the tables of frequency which are given below. In these the horizontal row of figures at the top of the table represents the groups involved, the vertical row on the left-hand margin represents the number of results written or answers copied. At the bottom of each table is given the average number of results written and the average number of answers copied for the various groups. On this basis is calculated the number of seconds required to write the five results or to copy the five answers of the group. The difference between these values for each group gives the number of seconds required to "think" the sums or products.

In order to test the validity of the results obtained, the average deviation is calculated for each of the averages; the standard deviation, the P. E. and the P. E. in seconds required to do five combinations are calculated for the maximum and minimum average deviations in the sums and products, and for the same groups in the answers copied. For the method by which these calculations were made the writer is indebted to Professor E. L. Thorndyke (44). For calculating the P. E. in seconds to do five combinations, the following proportion was used: *The P. E. in seconds required to do five sums or products or to copy five answers* (x) : *the number of seconds required to write five sums, &c. :: the P. E. in combinations written in 5 seconds : the number of combinations that are written in 5 seconds.* This will appear in the first calculation of the kind made as follows:

x sec. : 10.36 sec. :: .28/3 sums, &c. : 7.24/3 sums, &c., a solution of which gives .4 as the P. E. in seconds to write five sums. The true average will, therefore, be seen not to vary materially from the calculated average in any of the cases as far as time is concerned. For a full discussion of the reliability of averages, the reader is referred to Dr. Thorndyke's book, *Mental and Social Measurements,* already referred to above.

TABLE XVI.

SHOWING FOR A AND B CLASS PUPILS OF THE THIRD GRADE THE NUMBER OF SUMS IN THE VARIOUS GROUPS WRITTEN IN 15 SECONDS, THE NUMBER OF ANSWERS COPIED IN 5 SECONDS, AND GIVING THE TIME REQUIRED TO "THINK" THE ANSWERS TO THE SAME GROUPS.

PART 1. ANSWERS.

Groups	1		2		3		4		5		6		7		8		9	
Classes	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
Number written—																		
0	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
1	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
2	....	....	....	2	....	....	....	....	....	....	....	1	....	....	....	1	....	....
3	....	....	1	....	....	3	....	1	....	....	....	....	....	1	2	3	....	7
4	....	3	1	1	....	....	....	3	....	6	2	9	1	14	....	22	5	10
5	1	5	2	11	1	6	1	9	3	8	5	10	5	16	15	23	11	20
6	2	18	5	24	2	11	2	18	12	20	8	20	18	21	17	8	22	16
7	6	21	14	19	9	16	8	12	10	7	22	22	24	10	20	8	22	5
8	14	11	15	8	12	14	15	11	12	14	14	3	8	8	11	3	10	15
9	14	2	17	2	15	14	15	11	16	5	11	2	11	....	5	5	7	....
10	10	5	4	4	10	3	13	4	6	2	3	2	2	....	5	....	2	....
11	10	6	10	....	12	1	9	3	6	5	3	3	....	2	2	....	....	....
12	5	2	1	1	6	5	3	1	4	3	3	....	2	....	....	....	1	....
13	4	....	2	1	4	....	5	....	2	2	2	....	8	1	2	....	....	....
14	4	....	1	....	2	....	....	....	2	....	5	1	1	....	1	....	....	....
15	10	....	7	....	....	....	9	....	7	1	2	....	....	....	....	....	....	....
16	....	....	....	....	7	....	....	....	....	....	....	....	....	....	....	....	....	....
Averages	10.3	7.4	9.1	6.7	10.1	7.6	9.9	7.2	9.3	7.4	8.4	6.4	7.9	5.9	7.1	5.3	5.7	5.6
No. seconds to copy 5 answers	2.4	3.4	2.7	3.7	2.5	3.3	2.5	3.5	2.7	3.4	3.0	3.9	3.2	4.2	3.5	4.7	3.7	4.4
Average deviation	2.2	1.5	2.0	1.3	2.0	1.5	2.0	1.5	2.0	2.0	2.0	1.4	1.8	1.2	1.5	1.2	1.2	1.3
Standard deviation	....	....	....	....	....	....	....	....	....	....	2.63	....	....	....	....	1.58	....	....
P. E.	....	....	....	....	....	....	....	....	....	....	0.197	....	....	....	....	0.126	....	....
P. E. in seconds to write 5 answers	....	....	....	....	....	....	....	....	....	....	0.07	....	....	....	....	0.112	....	....

TABLE XVI.—PART 2. SUMS.

Groups	1		2		3		4		5		6		7		8		9	
Classes	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
Number written—																		
0	....	....	....	....	....	1	....	....	....	....	....	....	....	2	....	2	2	2
1	....	....	....	....	....	....	....	1	....	....	1	....	1	6	1	5	3	9
2	....	....	....	1	1	....	....	....	4	2	7	2	5	6	7	14	11	13
3	....	2	2	1	2	3	3	8	3	4	4	7	13	16	20	14	16	15
4	2	....	....	1	1	5	5	5	1	7	6	9	15	14	10	11	10	10
5	....	2	3	1	4	8	5	10	8	19	11	20	12	11	10	14	18	10
6	4	4	1	6	6	8	7	12	8	17	12	14	10	11	12	10	6	8
7	5	6	3	6	7	11	5	11	18	16	6	12	5	4	3	2	6	2
8	4	6	9	12	9	14	11	13	10	1	8	3	3	1	5	1	....	1
9	5	6	2	13	9	11	10	7	9	3	5	5	5	....	5	....	4	2
10	6	10	6	16	6	7	10	2	3	2	6	1	2	1	2	....	1	....
11	10	10	16	11	9	2	4	2	1	1	2	....	1	1	2	....	2	....
12	10	10	9	2	5	1	4	....	2	1	3	....	1	....	2	....	....	....
13	10	4	6	2	7	2	3	1	1	....	1	....	4	....	....	....	....	1
14	8	2	4	....	1	....	7	1	4	....	1	....	1	....	....	....	....	....
15	15	11	19	1	13	....	6	....	8	....	7	....	2	....	1	....	1	....
16	1	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
(1)	11.5	10.4	11.5	8.9	9.9	7.3	9.2	6.7	8.1	5.9	7.2	5.6	5.6	4.1	5.3	3.7	4.4	3.7
(2)	6.5	7.2	6.5	8.4	7.6	10.3	8.2	11.2	9.2	12.7	10.4	13.4	12.7	18.2	14.1	20.2	17.0	20.2
(3)	2.4	3.4	2.7	3.7	2.5	3.3	2.5	3.5	2.7	3.4	3.0	3.9	3.2	4.2	3.5	4.7	3.7	4.4
(4)	4.1	3.8	3.8	4.7	5.1	7.0	5.7	7.7	6.5	9.3	7.4	9.5	9.5	1.4	10.6	15.5	13.3	15.8
(5)	2.5	2.5	2.5	1.7	2.9	1.8	2.7	1.9	2.6	1.4	3.0	1.4	2.5	1.5	1.9	1.3	1.9	1.6
Standard deviation	....	....	....	....	....	....	....	....	....	....	3.32	....	....	....	....	1.79	....	....
P. E. in combinations	....	....	....	....	....	....	....	....	....	....	0.28	....	....	....	....	0.14	....	....
P. E. in seconds	....	....	....	....	....	....	....	....	....	....	0.4	....	....	....	....	0.76	....	....

NOTE.—(1) Average number of sums in 15 seconds.
(2) Number of seconds required to do 5 combinations.
(3) Number of seconds required to copy 5 answers, Table XVI, 1.
(4) Number of seconds required to "think" 5 combinations.
(5) Average deviation in number of combinations.

TABLE XVII.

SHOWING FOR A AND B CLASS PUPILS OF THE FOURTH GRADE THE NUMBER OF SUMS IN THE VARIOUS GROUPS WRITTEN IN 10 SECONDS, THE NUMBER OF ANSWERS COPIED IN 5 SECONDS, AND GIVING THE TIME REQUIRED TO "THINK" THE ANSWERS TO THE SAME GROUPS.

PART 1. ANSWERS.

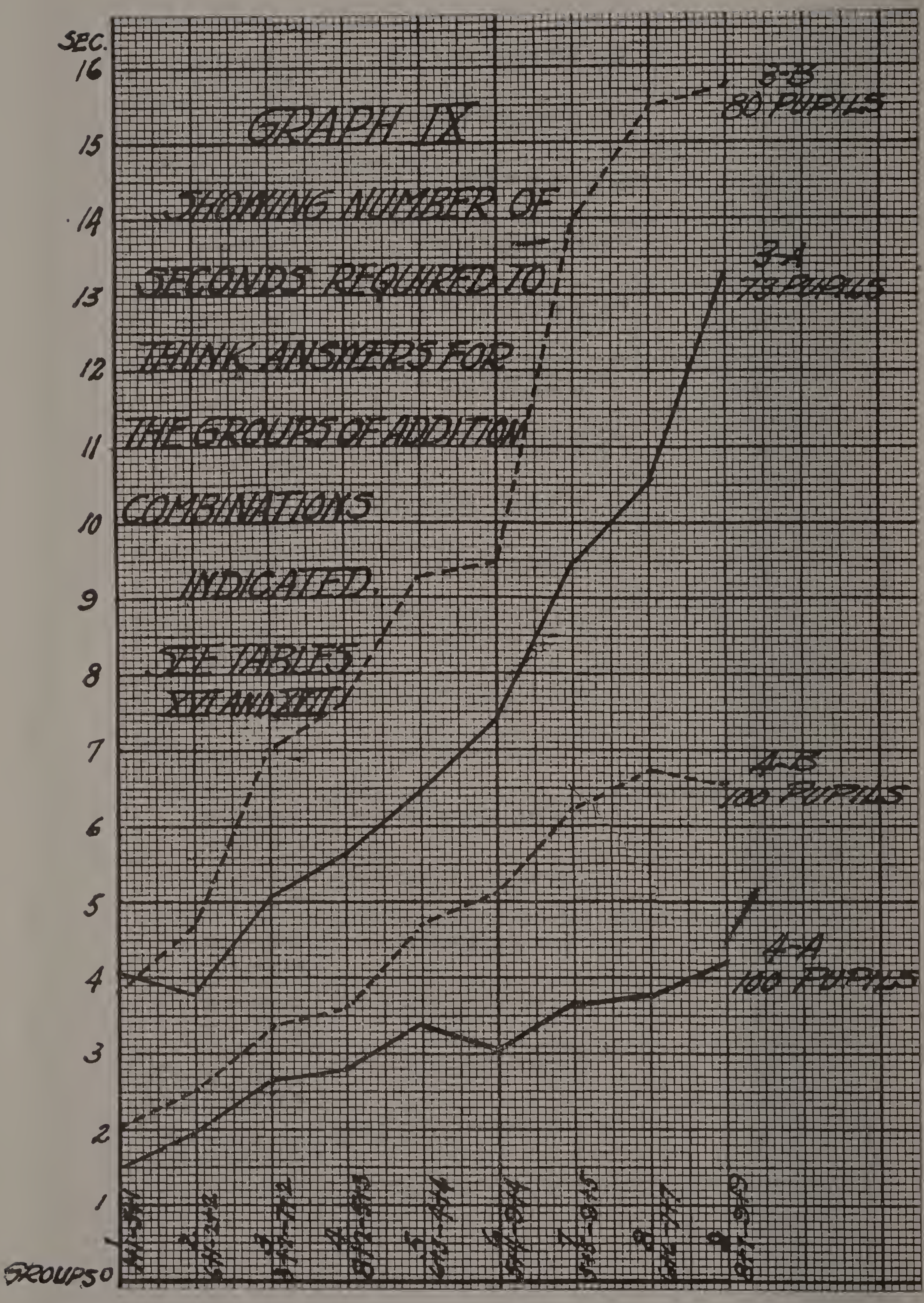
Groups	1		2		3		4		5		6		7		8		9	
Classes	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
Number written—																		
0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
3	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
4	.....	.....	.....	.....	.....	1	.....	.....	1	.....	.....	1	.....	.....	.....	2	.....	.....
5	.....	.....	.....	1	.....	.....	.....	2	.....	4	.....	8	1	10	12	13	4	16
6	3	11	3	12	1	7	1	6	2	16	12	26	21	37	32	38	21	41
7	6	24	5	23	4	14	7	20	6	32	22	44	34	38	24	38	26	29
8	17	23	17	36	16	26	16	38	24	27	31	10	21	10	19	7	36	12
9	22	26	29	13	20	32	19	18	23	12	16	9	13	4	10	2	13	2
10	15	8	23	8	19	15	26	12	20	6	10	1	7	1	3	.....	.....	.....
11	15	6	13	4	20	3	17	4	13	3	8	1	2	.....	.....	.....	.....	.....
12	12	1	7	1	12	2	9	.....	9	.....	.....	.....	1	.....	.....	.....	.....	.....
13	2	1	3	2	4	.....	5	.....	2	.....	.....	.....	.....	.....	.....	.....	.....	.....
14	2	.....	.....	.....	4	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
15	6	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
(1)	9.6	8.2	9.5	8.0	10.0	8.5	9.8	8.2	9.1	7.6	8.1	6.9	7.6	6.6	6.9	6.4	7.3	6.4
(2)	2.6	3.0	2.6	3.1	2.5	2.9	2.5	3.0	2.7	3.3	3.0	3.6	3.3	3.8	3.8	3.9	3.4	3.9
(3)	1.7	1.2	1.2	1.1	1.4	1.1	1.3	1.0	1.3	1.1	1.1	0.8	1.1	0.8	1.0	0.8	0.9	0.8
(4)	2.2	1.5	.....	.....	.....	.....	1.5	1.3	1.6	1.4	.....	.....	1.3	1.0	1.3	0.94	.....	.....
(5)	0.5	0.1	.....	.....	.....	.....	0.1	0.1	0.1	0.1	.....	.....	0.1	0.07	0.09	0.07	.....	.....
(6)	0.04	0.04	.....	.....	.....	.....	0.03	0.03	0.03	0.05	.....	.....	0.04	0.04	0.04	0.04	.....	.....

NOTE.—(1) Average number of answers in 5 seconds.
(2) Number of seconds required to copy 5 answers.
(3) Average deviation in number of combinations.
(4) Standard deviation in number of combinations.
(5) P. E. in number of combinations.
(6) P. E. in seconds to write 5 answers.

TABLE XVII.—PART 2. SUMS.

Groups	1		2		3		4		5		6		7		8		9	
Classes	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
Number written—																		
0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	2	.....	.....
2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	2	.....	2	2	10	1	10
3	.....	.....	.....	.....	.....	.....	.....	1	.....	7	.....	10	2	17	5	21	8	21
4	.....	.....	.....	.....	.....	2	.....	7	1	10	7	20	11	26	9	17	12	20
5	.....	.....	.....	1	5	9	8	10	8	21	8	14	15	21	17	16	12	10
6	.....	2	2	7	4	11	3	19	9	19	15	20	15	15	22	16	19	23
7	2	9	4	16	10	21	16	23	17	16	12	20	13	12	14	9	21	27
8	5	16	9	14	13	16	11	8	26	16	10	4	13	3	8	5	9	4
9	7	22	21	28	13	17	11	12	11	6	16	6	14	2	14	4	4	4
10	12	16	17	20	18	12	16	8	17	2	17	4	11	2	5	.....	8	1
11	13	12	15	9	12	5	18	7	6	2	10	.....	3	.....	4	.....	2	.....
12	15	10	10	3	15	4	8	3	4	.....	3	.....	3	.....	.....	.....	3	.....
13	15	7	10	1	6	2	9	1	1	.....	2	.....	.....	.....	.....	.....	1	.....
14	7	3	3	1	2	1	.....	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
15	24	2	9	.....	2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
(1)	12.1	9.8	10.6	8.8	9.7	7.9	9.3	7.5	8.2	6.2	8.1	5.7	7.1	5.0	6.6	4.7	6.5	4.8
(2)	4.1	5.1	4.7	5.6	5.2	6.3	5.4	6.7	6.1	8.0	6.2	8.8	6.99	10.0	7.6	10.7	7.5	10.4
(3)	2.6	3.0	2.6	3.1	2.5	2.9	2.5	3.0	2.7	3.3	3.0	3.6	3.3	3.8	3.8	3.9	3.4	3.9
(4)	1.5	2.1	2.1	2.5	2.7	3.4	2.9	3.7	3.4	4.7	3.2	5.2	3.6	6.2	3.8	6.8	4.1	6.5
(5)	2.1	1.7	1.8	1.5	1.9	1.6	1.9	2.0	1.5	1.5	2.0	1.6	1.9	1.3	1.7	1.6	1.8	1.6
(6)	2.3	2.1	.....	.....	.....	.....	2.3	2.3	1.8	1.9	.....	.....	2.2	1.7	.....	.....	.....	.....
(7)	0.15	0.14	.....	.....	.....	.....	0.15	0.15	0.12	0.13	.....	.....	0.15	0.11	.....	.....	.....	.....
(8)	0.052	0.073	.....	.....	.....	.....	0.09	0.14	0.09	0.16	.....	.....	0.15	0.226	.....	.....	.....	.....

NOTE.—(1) Average number of sums in 10 seconds.
(2) Number of seconds required to do 5 combinations.
(3) Number of seconds required to copy 5 answers.
(4) Number of seconds required to "think" 5 combinations.
(5) Average deviation in number of combinations.
(6) Standard deviation in number of combinations.
(7) P. E. in number of combinations.
(8) P. E. in seconds to do five combinations.


GRAPH IX
SHOWING NUMBER OF
SECONDS REQUIRED TO
THINK ANSWERS FOR
THE GROUPS OF ADDITION
COMBINATIONS
INDICATED.
SEE TABLES
SEC.
GROUPS
3-B
80 PUPILS
3-A
75 PUPILS
4-B
100 PUPILS
4-A
100 PUPILS

Discussion.—A study of Graph IX. demonstrates the growth in skill from the 3–B to the 4–A class. It also shows a rapid slowing up on the work as the groups contain larger numbers. From the standpoint of skill, we can, therefore, say that difficulty in addition increases as the magnitude of the numbers increase—that is, within the limits of the forty-five elementary facts.

Tables XVI and XVII will sufficiently demonstrate the method pursued in the study of facility in the use of the combinations to justify the omission of the details of Tables XVIII and XIX except as to the averages, &c., which are given in detail below. The original tables of frequency will be found on file with other details of this investigation which space will not allow us to give here.

TABLE XVIII.

SHOWING FOR A AND B CLASS THIRD GRADE PUPILS THE NUMBER OF PRODUCTS WRITTEN IN 10–25 SECONDS IN THE VARIOUS MULTIPLICATION GROUPS, THE NUMBER OF ANSWERS COPIED IN 5 SECONDS, AND GIVING THE TIME REQUIRED TO "THINK" THE PRODUCTS FOR THE SAME GROUPS.

1. ANSWERS.

Groups	(1)	(2)	(3)	(4)	(5)	(6)
1—A	9.04	2.76	1.6	2.07	0.15	0.05
B	7.73	3.23	1.5	1.98	0.19	0.08
2—A	9.73	2.76	1.6	....	....	....
B	7.92	3.16	1.61	....	....	....
3—A	8.93	2.8	1.98	....	....	....
B	7.23	3.45	1.61	....	....	....
4—A	7.54	3.31 (2.65)	1.46	....	....	....
B	5.94	4.21 (3.36)	1.13	....	....	....
5—A	5.76	4.37	1.59	....	....	....
B	5.55	4.50	1.00	....	....	....
6—A	6.32	3.96	1.22	1.87	0.14—	0.09—
B	4.8	5.21	0.86	1.16	0.11+	0.12
7—A	5.98	4.18	1.05	....	....	....
B	4.45	5.62	0.92	....	....	....
8—A	5.65	4.42	1.00	1.45	0.11—	0.085
B	4.73	5.28	0.98	1.52	0.15—	0.16+
9—A	6.01	4.16	1.3	....	....	....
B	4.45	5.62	1.06	....	....	....
10—A	5.56	4.5	1.44	2.15	0.18+	0.15—
B	3.8	6.58	0.7	0.91	0.09—	0.15
11—A	5.51	4.54	1.00	1.29	0.10—	0.08
B	4.57	5.47	0.72	1.07	0.10+	0.12+
12—A	6.07	4.12	1.46	....	....	....
B	4.43	5.64	0.74	....	....	....
13—A	5.88	4.25	1.2	1.65	0.12+	0.09—
B	4.37	5.72	0.96	1.26	0.12+	0.16—
14—A	5.63	4.44	1.07	1.59	0.12—	0.095
B	4.39	5.72	0.78	1.07	0.10+	0.13+

TABLE XVIII.—*Continued.*

SHOWING FOR A AND B CLASS THIRD GRADE PUPILS THE NUMBER OF PRODUCTS WRITTEN IN 10–25 SECONDS IN THE VARIOUS MULTIPLICATION GROUPS, THE NUMBER OF ANSWERS COPIED IN 5 SECONDS, AND GIVING THE TIME REQUIRED TO "THINK" THE PRODUCTS FOR THE SAME GROUPS.

1. ANSWERS.

Groups	(1)	(2)	(3)	(4)	(5)	(6)
15—A	5.26	4.75	1.05	....	....	....
B	4.16	6.01	0.7	....	....	....
16—A	5.06	4.94 (3.95)	1.12	1.59	0.12—	0.12—
B	4.04	6.19 (4.95)	0.82	1.32	0.13—	0.19+

NOTE.—(1) Average number of answers copied in 5 seconds.
(2) Average number of seconds required to copy 5 answers.
(3) Average deviation from the average number of answers.
(4) Standard deviation.
(5) P. E. in number of answers copied.
(6) P. E. in seconds required to copy 5 answers of each group.

(The numbers in parenthesis for groups 4 and 16 indicate the time required to copy the four answers constituting the group.)

TABLE XIX.

PRODUCTS.

Groups ...	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1—A* ...	10.72	4.63	2.76	1.9	2.84	3.33	0.25	0.11—
B	9.47	7.92	3.23	4.7	3.4	3.97	0.38+	0.31—
2—A* ...	11.57	4.32	2.76	1.6	2.4	......	....	....
B	11.61	6.49	3.16	3.3	2.45	......	....	....
3—A	8.21	6.09	2.8	3.3	2.04	......	....	....
B	7.00	10.71	3.45	7.3	2.35	......	....	....
4—A	7.51	6.66	3.31	3.35	1.88	......	....	....
		(5.3)	(2.6)	(2.7)				
B	7.25	10.34	4.21	6.1	2.3	......	....	....
		(8.3)	(3.7)	(4.9)				
5—A	7.25	6.89	4.36	2.5	1.67	......	....	....
B	6.79	11.05	4.5	6.55	2.65	......	....	....
6—A	5.96	8.39	3.96	4.4	2.01	2.63	0.20	0.28—
B	5.61	13.37	5.21	8.2	2.27	3.1	0.28+	0.68
7—A	5.82	8.59	4.18	4.4	1.74	......	....	....
B	6.29	11.92	5.62	6.3	2.02	......	....	....
8—A	5.27	9.49	4.42	5.1	1.85	2.48	0.19—	0.34—
B	4.99	15.03	5.28	9.75	2.31	2.75	0.265	0.8—
9—A	6.56	7.64	4.16	3.5	1.98	......	....	....
B	6.16	12.17	5.62	6.55	2.15	......	....	....
10—A	8.19	9.16	4.5	4.7	1.88	2.3	0.17+	0.19+
B† ...	6.35	14.97	6.58	8.4	2.96	3.31	0.32—	0.47+
11—A	7.49	10.01	4.54	5.5	2.12	2.65	0.2—	0.27—
B† ...	7.18	17.41	5.47	11.9	3.33	4.19	0.39—	0.94+
12—A	7.4	10.14	4.12	6.0	1.93	......	....	....
B† ...	7.45	16.79	5.64	11.15	2.51	......	....	....
13—A	7.56	9.93	4.25	5.7	1.83	2.26	0.17—	0.22+
B† ...	7.8	16.03	5.72	11.3	3.41	4.08	0.39+	0.81—
14—A	6.9	10.87	4.44	6.4	2.11	2.65	0.2—	0.31+
B† ...	8.2	15.24	5.72	10.5	3.43	3.78	0.36+	0.67+
15—A	6.34	11.83	4.75	7.1	1.76	......	....	....
B† ...	8.08	15.47	6.01	9.5	2.80	......	....	....

TABLE XIX.—*Continued.*

PRODUCTS.

Groups ...	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
16—A	5.86	12.80 (10.2)	4.94 (3.9)	7.9 (6.3)	2.15	2.6—	0.195	0.43—
B† ...	6.7	18.66 (14.9)	6.19 (4.9)	12.5 (10.0)	2.61	3.06	0.29+	0.98+

NOTE.—No mark after class letter indicates that 15 seconds were given for each group. The * after the class letter shows that 10 seconds were allotted. The † after the class letter indicates that 25 seconds were given to the corresponding groups.

(1) Average number of products written.
(2) Number of seconds required to write 5 products.
(3) Number of seconds required to copy 5 answers. (From Table XVIII, 1.)
(4) Number of seconds required to "think" 5 combinations.
(5) Average deviation from average (1).
(6) Standard deviation.
(7) P. E. in number of combinations written in time allowed.
(8) P. E. in number of seconds required to write 5 combinations.

(The numbers in parenthesis for groups 4 and 16 indicate the time required to write the four products in the group.)

TABLE XX.

SHOWING FOR A AND B CLASS PUPILS OF THE FOURTH GRADE THE AVERAGE NUMBER OF ANSWERS COPIED IN 5 SECONDS, THE NUMBER OF SECONDS REQUIRED TO COPY FIVE ANSWERS, ETC.

ANSWERS.

Groups	(1)	(2)	(3)	(4)	(5)	(6)
1—A	10.88	2.3	1.78	2.1	0.142	0.03
B	8.94	2.8	1.22	1.7	0.115	0.036
2—A	10.45	2.4	1.51	....	....	....
B	8.71	2.87	0.84	....	....	....
3—A	9.38	2.67	1.58	....	....	....
B	7.98	3.13	0.96	....	....	....
4—A	7.6	3.19 (2.63)	1.27	....	....	....
B	6.87	3.64 (2.9)	0.72	....	....	....
5—A	7.03	3.55	1.39	....	....	....
B	5.83	4.29	0.70	....	....	....
6—A	6.66	3.75	1.13	1.44	0.097	0.054
B	6.00	4.17	0.66	0.94	0.063	0.044
7—A	5.92	4.22	1.03	....	....	....
B	5.23	4.78	0.61	....	....	....
8—A	5.93	4.22	0.89	1.1	0.074	0.052
B	5.20	4.81	0.78	0.95	0.064	0.059
9—A	5.80	4.31	1.00	....	....	....
B	5.07	4.93	0.68	....	....	....
10—A	5.47	4.57	1.19	1.46	0.098	0.1
B	4.98	5.02	0.61	0.88	0.059	0.06
11—A	6.05	4.13	1.09	1.39	0.094	0.064
B	5.08	4.92	0.56	0.81	0.054	0.052
12—A	6.20	4.03	1.09	....	....	....
B	5.01	5.00	0.58	....	....	....
13—A	6.21	4.02	1.00	1.26	0.085	0.055
B	5.05	4.95	0.69	0.88	0.059	0.057
14—A	5.96	4.19	0.93	1.25	0.084	0.059
B	4.89	5.11	0.62	0.81	0.054	0.056

TABLE XX.—*Continued.*

SHOWING FOR A AND B CLASS PUPILS OF THE FOURTH GRADE THE AVERAGE NUMBER OF ANSWERS COPIED IN 5 SECONDS, THE NUMBER OF SECONDS REQUIRED TO COPY FIVE ANSWERS, ETC.

ANSWERS.

Groups	(1)	(2)	(3)	(4)	(5)	(6)
15—A	5.67	4.41	1.08	....	....	....
B	4.72	5.3	0.67	....	....	....
16—A	5.23	4.78 (3.8)	1.00	1.15	0.078	0.071
B	4.23	5.91 (4.7)	0.62	0.80	0.054	0.075

NOTE.—For explanation, see note to Table XVIII, 1.

TABLE XXI.

SHOWING FOR A AND B CLASS PUPILS OF THE FOURTH GRADE THE NUMBER OF PRODUCTS OF THE VARIOUS MULTIPLICATION GROUPS WRITTEN IN 10 SECONDS, ALSO THE NUMBER OF SECONDS REQUIRED TO "THINK" THE FIVE (OR FOUR) COMBINATIONS OF EACH GROUP.

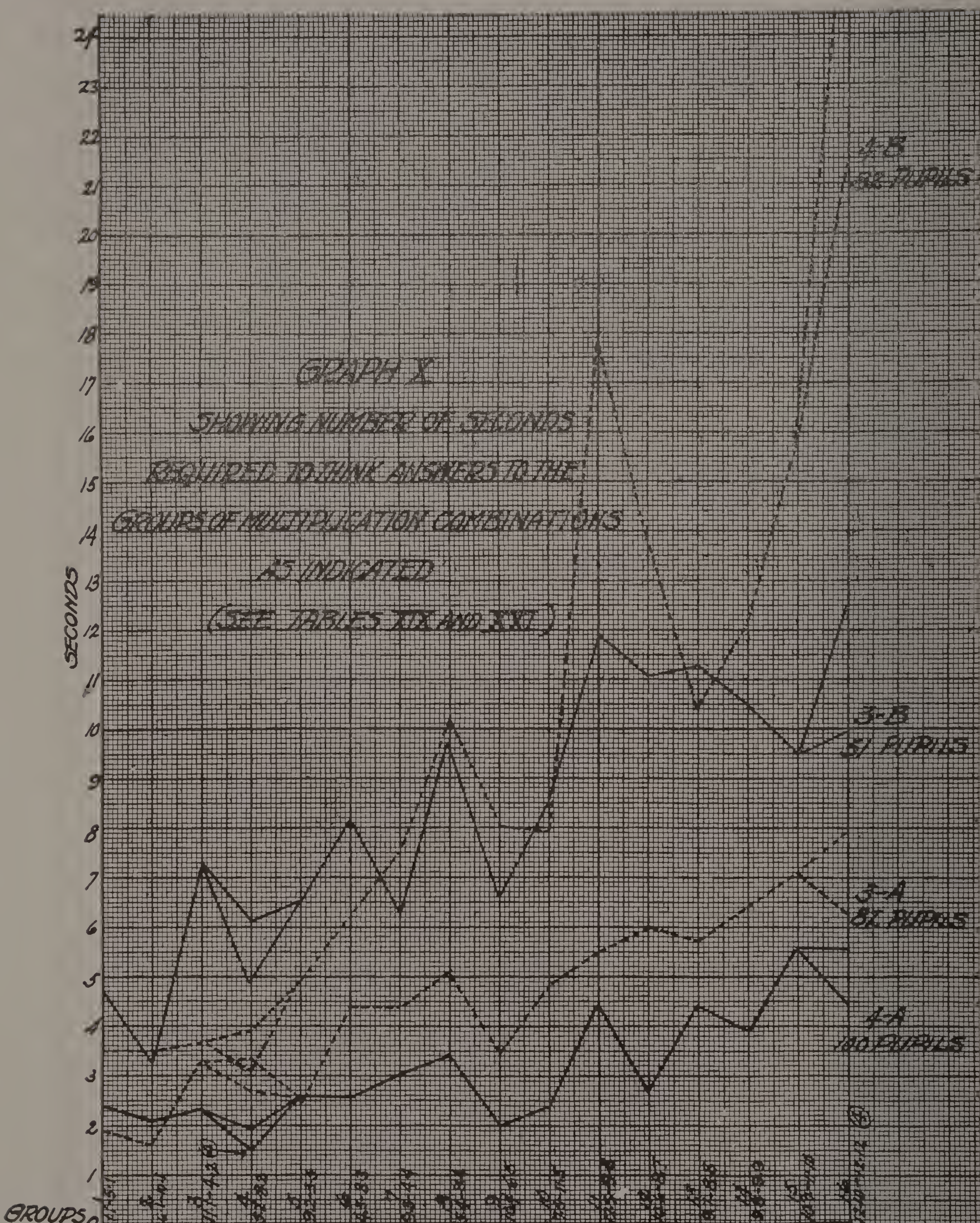
Groups	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1—A	10.66	4.69	2.3	2.4	2.8	3.07	0.207	0.092
B	7.94	6.3	2.8	3.5	2.2	2.63	0.177	0.114
2—A	11.16	4.48	2.4	2.1	2.56	....	....	....
B	7.81	6.4	2.87	3.5	2.75	....	....	....
3—A	10.16	4.92	2.67	2.3	2.1	....	....	....
B	7.35	6.8	3.13	3.7	1.8	....	....	....
4—A	9.75	5.13	3.19	1.9	1.6	....	....	....
		(4.1)	(2.6)	(1.5)				
B	6.65	7.5	3.64	3.9	1.5	....	....	....
		(6.0)	(2.9)	(3.1)				
5—A	8.18	6.11	3.55	2.6	1.7	....	....	....
B	5.46	9.16	4.29	4.9	1.4	....	....	....
6—A	7.89	6.34	3.75	2.6	1.3	1.55	0.105	0.084
B	4.86	10.4	4.17	6.2	1.4	1.7	0.115	0.24
7—A	6.99	7.2	4.22	3.0	1.5	....	....	....
B	4.07	12.28	4.78	7.5	1.44	....	....	....
8—A	6.58	7.6	4.22	3.4	1.9	2.074	0.139	0.16
B	3.34	14.97	4.81	10.2	1.5	1.84	0.124	0.56
9—A	7.77	6.43	4.31	2.1	1.4	....	....	....
B	3.92	13.01	4.93	8.1	1.3	....	....	....
10—A	7.12	7.02	4.57	2.4	1.7	2.06	0.139	0.137
B	3.88	12.88	5.02	7.9	1.2	1.46	0.098	0.325
11—A	5.84	8.56	4.13	4.4	1.7	2.16	0.146	0.215
B	2.2	22.73	4.92	17.8	1.2	1.37	0.092	0.95
12—A	6.46	7.74	5.03	2.7	1.7	....	....	....
B	2.66	18.8	5.0	13.8	1.3	....	....	....
13—A	5.91	8.46	4.02	4.4	1.8	2.2	0.148	0.212
B	3.26	15.34	4.95	10.4	1.6	1.7	0.114	0.536
14—A	6.19	8.3	4.19	3.9	1.7	2.02	0.148	0.192
B	2.90	17.24	5.11	12.1	0.9	1.45	0.098	0.571

TABLE XXI.—*Continued.*

SHOWING FOR A AND B CLASS PUPILS OF THE FOURTH GRADE THE NUMBER OF PRODUCTS OF THE VARIOUS MULTIPLICATION GROUPS WRITTEN IN 10 SECONDS, ALSO THE NUMBER OF SECONDS REQUIRED TO "THINK" THE FIVE (OR FOUR) COMBINATIONS OF EACH GROUP.

Groups	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
15—A	4.97	10.06	4.41	5.6	1.4	....	....	....
B	2.38	21.0	5.3	15.8	1.3	....	....	....
16—A	4.82	10.37 (8.3)	4.78 (3.8)	5.6 (4.5)	1.7	2.0	0.135	0.29
B	1.53	32.68 (26.0)	5.91 (4.7)	26.8 (21.4)	0.6	1.037	0.069	0.148

NOTE.—For explanation, see note to Table XIX.


GRAPH X
SHOWING NUMBER OF SECONDS
REQUIRED TO THINK ANSWERS TO THE
GROUPS OF MULTIPLICATION COMBINATIONS
AS INDICATED
(SEE TABLES XIX AND XXI)
SECONDS
GROUPS
4-B
52 PUPILS
3-B
51 PUPILS
3-A
82 PUPILS
4-A
180 PUPILS

Discussion.—In giving these multiplication tests the sixteen groups of facts were divided into two parts, the first part consisting of the first nine groups in their numerical order, and the second part of the last seven groups. These groups were given first in their regular order, then the second part was given first, and finally in the third test the order of the groups was reversed. This plan had for its object the equalizing of the effects of fatigue, distributing it as nearly as possible throughout the whole series.

A study of Graph X shows a tendency upward throughout the "curves" in somewhat the same manner as the "curve" of errors in Graph VI. The most remarkable thing about this figure is the irregularity of the "curve" for the 4–B class. In the natural course of events it should fall between the "curve" for 3–A and that for 4–A as in Graph IX. This curve is doubtless due to the fact at the time these tests were given the 4–B class had been engaged for some time in studying the addition and subtraction of simple fractions. The test caught the 3–B class drilling on the latter part of the tables, hence the increase of skill noted in that class from the 11th group on to the end.

The two values given in groups 4 and 16 are due to calculation on the basis of both 4 and 5 combinations, the 5 combination value being the higher.

An analysis of the four "curves" shows that groups 3, 6, 8, 11, and 16 present a marked degree of difficulty from the standpoint of skill, as compared with the groups adjacent to them. Groups 2, 4, 7, 9, and 12 are correspondingly easy. The fact that group 1 shows higher difficulty than group 2 is doubtless due to the errors of process made in 1×1, which were in these tests counted as errors. Arranged then according to their relative difficulty, beginning with the easiest, the groups would take the following order: 2, 1, 4, 3, 5, 9, 10, 7, 6, 8, 12, 13, 14, 15, 11, 16.

A comparison of Graph X with the per cent. of errors part of Graph VI shows that skill in the manipulation of the combinations as represented by time is in inverse ratio to the number of errors made, the high points of each "curve" being in practically the same relative positions. The like is also observed in the addition by a comparison of Graphs V and IX. In other words, the

groups showing the greatest skill in handling are the groups in which the fewest errors appear. Rapidity in this case goes with fewness of errors.

XVIII. CONCLUSIONS.

1. In the process of learning each group of combinations power is developed whereby subsequent groups are more easily mastered.

2. Girls learn the addition combinations with less difficulty than boys by about 10%, and make 10% fewer errors throughout the process.

3. Girls learn the multiplication combinations with about 25% less difficulty than boys, and make about 13% fewer errors in the process of mastery, but boys retain 2.7% better than girls when the whole number of products is considered, excelling especially in the larger ones, while girls excel slightly in retaining the smaller products.

4. In the process of teaching the addition facts fewer errors appear in the double numbers even when equal opportunity for making errors is provided, the fewest appearing in $5 + 5$ and the greatest number in $8 + 8$.

5. In teaching multiplication the fewest errors appear in the 10's and 11's and the next fewest in the doubles after 3×3.

6. Combinations in the latter part of the series in both addition and multiplication are more readily learned, but they are also more easily forgotten.

7. From the standpoint of skill in manipulation, difficulty in addition increases with the magnitude of the numbers.

8. The groups showing least skill for their manipulation are the same as those showing the greatest number of errors. This shows close correlation between speed and accuracy, greater speed going along with greater accuracy.

9. The story of relative difficulty for addition facts is told in detail by Graphs V and VII.

10. The story of relative difficulty for multiplication combinations is told in detail by Graphs VI and VIII.

XIX. SUGGESTIONS GROWING OUT OF THE INVESTIGATION.

1. The combinations in both addition and multiplication should be taught in smaller groups than the ordinary tables. Five combinations per group is suggested as best, giving as it does an average of one new combination for each school day of the week.

2. In a group of five, or any other number of facts to be learned, it should be remembered that the first and the last position are points of vantage, and that the hardest combinations of a given group should be placed at these two points, and the easier ones in the middle.

3. Subtraction should be taught simultaneously with addition, and as an additive process.

4. Division should be taught simultaneously with multiplication.

5. It is not economical to begin *drills* in addition facts before the beginning of the second half of the first school year.

6. The multiplication combinations may profitably be begun in the second half of the second school year. They may all be taught in one term, if that term falls in the second half of the year, but it is not advisable to attempt beyond 10×10 in this term.

7. In both addition and multiplication, combinations should be taught in both direct and reverse forms simultaneously; as, $2 + 3 = 5$, $3 + 2 = 5$; or $6 \times 7 = 42$, $7 \times 6 = 42$.

8. No scheme of association, no number plays or games which have not in them as their predominant factor the idea and essence of drill, can take the place of systematic and persistent repetition as a means of making permanent and automatic the correct reactions to the combinations in addition and multiplication.

Education has too long been philosophical. Every phase of the subjects taught in the schools is fairly bristling with unsolved problems—problems that are "settled" to-day in a dogmatic way and unsettled to-morrow in a dogmatic way, the stability of the decisions depending upon the position and the persistence and the eloquence of the individual who renders the

dictum. While the day of dictum is not over, the dawn of a new order of things in educational procedure has already begun. The only way in which pedagogy may write its name permanently among the sciences is by basing its conclusions upon the incontrovertible evidence of scientific investigation. This task is no easy one. It is too big for the individual investigator. He can at best only point the way. Such a study as this may posit conclusions only tentatively. There should be in every large system of education a department for the investigation of the problems of the school room. This department might hope to make its investigations sufficiently extensive to render its conclusions universally valid. Every large industrial establishment maintains such a department. It should be possible in the work of education.

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